

Orbifold melting crystal models and reductions of Toda hierarchy

Kanehisa Takasaki*

Department of Mathematics, Kinki University

3-4-1 Kowakae, Higashi-Osaka, Osaka 577-8502, Japan

Abstract

Orbifold generalizations of the ordinary and modified melting crystal models are introduced. They are labelled by a pair a, b of positive integers, and geometrically related to $\mathbb{Z}_a \times \mathbb{Z}_b$ orbifolds of local \mathbb{CP}^1 geometry of the $\mathcal{O}(0) \oplus \mathcal{O}(-2)$ and $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ types. The partition functions have a fermionic expression in terms of charged free fermions. With the aid of shift symmetries in a fermionic realization of the quantum torus algebra, one can convert these partition functions to tau functions of the 2D Toda hierarchy. The powers L^a, \bar{L}^{-b} of the associated Lax operators turn out to take a special factorized form that defines a reduction of the 2D Toda hierarchy. The reduced integrable hierarchy for the orbifold version of the ordinary melting crystal model is the bi-graded Toda hierarchy of bi-degree (a, b) . That of the orbifold version of the modified melting crystal model is the rational reduction of bi-degree (a, b) . This result seems to be in accord with recent work of Brini et al. on a mirror description of the genus-zero Gromov-Witten theory on a $\mathbb{Z}_a \times \mathbb{Z}_b$ orbifold of the resolved conifold.

2010 Mathematics Subject Classification: 17B65, 35Q55, 81T30, 82B20

Key words: melting crystal, orbifold model, free fermion, quantum torus, shift symmetry, Toda hierarchy, rational reduction

*E-mail: takasaki@math.h.kyoto-u.ac.jp

1 Introduction

Recently, we extended our previous work [1, 2] on the integrable structure of a modified melting crystal model to open string amplitudes of topological string theory on a generalized conifold [3]. The modified melting crystal model is essentially a reformulation of open string amplitudes, or local Gromov-Witten invariants, on the resolved conifold. Brini [4] conjectured, and partially proved at low orders of genus expansion, a mirror-theoretic correspondence between the local Gromov-Witten theory of the resolved conifold and the Ablowitz-Ladik hierarchy [5, 6]. In the course of refining this observation, Brini and his collaborators [7] reformulated this integrable hierarchy as a special “rational reduction” of the 2D Toda hierarchy [8, 9]. We considered this issue from a different route once developed for the ordinary melting crystal model [10, 11]. Technical clues therein are —

- (i) a fermionic expression of the partition function,
- (ii) an associated fermionic realization of the quantum torus algebra,
- (iii) algebraic relations called “shift symmetries” in this algebra,
- (iv) a matrix factorization problem that solves the initial value problem in the Lax formalism.

Armed with these tools, we proved that the Ablowitz-Ladik hierarchy indeed underlies the modified melting crystal model [1, 2], and extended this result to a generalized conifold [3]. This generalized conifold is of the *bubble* type, namely, its web diagram is a linear chain of repetition of the web diagram of the resolved conifold. The relevant integrable hierarchy in this case is a kind of generalized Ablowitz-Ladik hierarchy that is realized as a reduction of the 2D Toda hierarchy.

In this paper, we present yet another extension of our study on the melting crystal models. This work is motivated by a very recent paper of Brini et al. [12] on a general scheme of rational reductions of the 2D Toda hierarchy and its application to the mirror theory of a toric Calabi-Yau *orbifold*. They proposed a large class of rational reductions, and proved that the dispersionless limit of a particular reduction captures the genus-zero Gromov-Witten theory of a $\mathbb{Z}_a \times \mathbb{Z}_b$ orbifold of the resolved conifold. Remarkably, the reduced integrable hierarchy therein, too, is a generalized Ablowitz-Ladik hierarchy, but totally different from ours. In the case of the orbifold, the a -th and b -th powers of the Lax operators L, \bar{L}^{-1} of the 2D Toda hierarchy are expected to take the “rational” form

$$L^a = BC^{-1}, \quad \bar{L}^{-b} = DCB^{-1},$$

where D is a non-zero constant, normalized to 1 in the setting of Brini et al., and B and C are difference operators of the form

$$\begin{aligned} B &= e^{a\partial_s} + \beta_1(s)e^{(a-1)\partial_s} + \cdots + \beta_a(s), \\ C &= 1 + \gamma_1(s)e^{-\partial_s} + \cdots + \gamma_b(s)e^{-b\partial_s} \end{aligned}$$

on the 1D lattice with coordinate s . In the case of the bubble type [3], the Lax operators themselves take the rational form

$$L = Be^{(1-N)\partial_s}C^{-1}, \quad \bar{L}^{-1} = DCe^{(N-1)\partial_s}B^{-1},$$

where N is the number of repetition of the conifold diagram, D is a non-zero constant, and B and C are difference operators of the form

$$\begin{aligned} B &= e^{N\partial_s} + \beta_1(s)e^{(N-1)\partial_s} + \cdots + \beta_N(s), \\ C &= 1 + \gamma_1(s)e^{-\partial_s} + \cdots + \gamma_N(s)e^{-N\partial_s}. \end{aligned}$$

One of our goals is to derive the former — “the rational reduction of bi-degree (a, b) ” in the terminology of Brini et al. [12] — from a generalized melting crystal model.

Such a melting crystal model can be found in the work of Bryan et al. [13] on an orbifold version of the method of topological vertex [14]. They illustrated their method with two examples. One of them is local $\mathbb{CP}_{a,b}^1$ geometry, namely, the total space of the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over a $\mathbb{Z}_a \times \mathbb{Z}_b$ orbifold $\mathbb{CP}_{a,b}^1$ of \mathbb{CP}^1 . This, too, is an orbifold generalization of the resolved conifold. The orbifold topological vertex construction gives a generating function of Donaldson-Thomas invariants of this Calabi-Yau threefold. Its main part may be thought of as a generalized melting crystal model. We call this model *an orbifold melting crystal model*. In the non-orbifold case where $a = b = 1$, this model reduces to the modified melting crystal model. We shall show that this orbifold version of the modified melting crystal model is indeed accompanied by the rational reduction of bi-degree (a, b) of the 2D Toda hierarchy.

In the same sense, one can construct an orbifold version of the ordinary melting crystal model. Geometrically, this amounts to local $\mathbb{CP}_{a,b}^1$ geometry of the $\mathcal{O}(0) \oplus \mathcal{O}(-2)$ type. We shall show, as another goal, that this orbifold model corresponds to the bi-graded Toda hierarchy of bi-degree (a, b) [15, 16]. This integrable hierarchy is a reduction of the 2D Toda hierarchy for which the Lax operators satisfy the reduction condition

$$L^a = D^{-1}\bar{L}^{-b},$$

where D is a non-zero constant that is usually normalized to 1. Both sides of this condition become a difference operator of the form

$$\mathfrak{L} = e^{a\partial_s} + \alpha_1(s)e^{(a-1)\partial_s} + \cdots + \alpha_{a+b}(s)e^{-b\partial_s}.$$

When $a = b = 1$, \mathfrak{L} reduces to the well known Lax operator of the 1D Toda hierarchy. Just like the 1D Toda hierarchy [17, 18, 19], the bi-graded Toda hierarchy can be extended by extra logarithmic flows [16], and has been applied to the Gromov-Witten theory of an orbifold of \mathbb{CP}^1 [20, 21]. In the case of the orbifold melting crystal model, both sides of the reduction condition are further factorized as

$$L^a = D^{-1}\bar{L}^{-b} = BC,$$

where B and C are difference operators of the same form as those in the rational reduction of bi-degree (a, b) . Our previous melting crystal model with “two q -parameters” [22]¹ turns out to be a special case of this orbifold model.

The most prominent characteristic of the orbifold cases is that the powers L^a, \bar{L}^{-b} of L, \bar{L}^{-1} rather than L, \bar{L}^{-1} themselves show up in the reduction condition. This affects the correspondence between the coupling constants t_k, \bar{t}_k of the models and the time variables of the 2D Toda hierarchy as well. From a technical point of view, this characteristic is related to *fractional framing factors* in the orbifold topological vertex construction [13]. We shall encounter their avatars in our calculations. They are responsible for the emergence of the powers L^a, \bar{L}^{-b} of the Lax operators.

This paper is organized as follows. The orbifold version of the ordinary model is referred to as *the first orbifold model*. The orbifold version of the modified model is distinguished as *the second orbifold model*. Throughout the paper, these models are treated in a fully parallel way. Section 2 is devoted to formulation of these models. Sections 3 and 4 are focused on implications of the fermionic expression of the partition functions. Section 5 and 6 present the aforementioned results on the Lax operators. Let us explain the contents in more detail.

In Section 2, these two orbifold models are defined as models of random partitions, and translated to the language of charged free fermions. The Boltzmann weights of the partition functions are built from special values of the infinite-variable Schur functions $s_\lambda(\mathbf{x})$, $\mathbf{x} = (x_1, x_2, \dots)$. These weights are deformed by external potentials $\Phi_k(\lambda, s)$, where $k = 1, 2, \dots$ in the first orbifold model and $k = \pm 1, \pm 2, \dots$ in the second orbifold model. Note that

¹Note that this model is not directly related to the so called “refinement”.

these potentials depend on the lattice coordinate s as well. The partition function $Z_{a,b}(s, \mathbf{t})$ of the first orbifold model is a function of s and a set of coupling constants $\mathbf{t} = (t_1, t_2, \dots)$. The partition function $Z'_{a,b}(s, \mathbf{t}, \bar{\mathbf{t}})$ of the second orbifold model depends on s and two sets of coupling constants $\mathbf{t} = (t_1, t_2, \dots)$ and $\bar{\mathbf{t}} = (\bar{t}_1, \bar{t}_2, \dots)$.

In Sections 3 and 4, these partition functions are converted to tau functions of the 2D Toda hierarchy. This is a somewhat lengthy and complicated generalization of the calculations for the ordinary [10, 11] and modified [1, 2] melting crystal models. We start from the fermionic expression of the partition functions that does not look like tau functions. This expression contains the generators $e^{H(\mathbf{t})}$ and $e^{\bar{H}(\bar{\mathbf{t}})}$ of time evolutions. $H(\mathbf{t})$ and $\bar{H}(\bar{\mathbf{t}})$ are linear combinations of fermion bilinears H_k and H_{-k} that are elements of a fermionic realization of the quantum torus algebra. We use the aforementioned shift symmetries of this algebra to transform $H_{\pm k}$'s to elements of the $U(1)$ current algebra, namely, infinitesimal generators of time evolutions of the 2D Toda hierarchy. By these calculations, the partition functions turn out to be, up to a relatively simple prefactor, tau functions of the 2D Toda hierarchy (Theorems 1 and 2).

In Sections 5 and 6, these special solutions of the 2D Toda hierarchy are re-interpreted in the Lax formalism. To this end, we make use of a matrix factorization problem that determines the dressing operators W, \bar{W} and the Lax operators L, \bar{L} . (These operators are translated to $\mathbb{Z} \times \mathbb{Z}$ matrices in advance.) Fortunately, the factorization problem in the present setting can be solved explicitly at the initial time $\mathbf{t} = \bar{\mathbf{t}} = \mathbf{0}$. This enables us, after some algebra, to derive the factorized form of L^a and \bar{L}^{-b} at the initial time. Since the factorized form is known to be preserved by all time evolutions of the 2D Toda hierarchy, this is enough to deduce the conclusion (Theorem 3).

2 Orbifold melting crystal models

2.1 Partition functions

The ordinary melting crystal model [23, 24] and its resolved conifold version [25, 26] are defined, respectively, by the partition functions

$$Z = \sum_{\lambda \in \mathcal{P}} s_\lambda (q^{-\rho})^2 Q^{|\lambda|}, \quad (2.1)$$

$$Z' = \sum_{\lambda \in \mathcal{P}} s_\lambda (q^{-\rho}) s_{\iota\lambda} (q^{-\rho}) Q^{|\lambda|}, \quad (2.2)$$

where \mathcal{P} denotes the set of all partitions $\lambda = (\lambda_i)_{i=1}^\infty$, ${}^t\lambda$ the conjugate (or transposed) partition of λ , $|\lambda|$ the sum of all parts λ_i , and $s_\lambda(q^\rho)$ the special value of the infinite-variable Schur function $s_\lambda(\mathbf{x})$, $\mathbf{x} = (x_1, x_2, \dots)$ [27], at

$$q^{-\rho} = (q^{1/2}, q^{3/2}, \dots, q^{n-1/2}, \dots).$$

The orbifold models are obtained by replacing the Boltzmann weights as

$$s_\lambda(q^{-\rho})^2 \rightarrow s_\lambda(p_1 q^{-\rho}, \dots, p_a q^{-\rho}) s_\lambda(r_1 q^{-\rho}, \dots, r_b q^{-\rho})$$

and

$$s_\lambda(q^{-\rho}) s_{{}^t\lambda}(q^{-\rho}) \rightarrow s_\lambda(p_1 q^{-\rho}, \dots, p_a q^{-\rho}) s_{{}^t\lambda}(r_1 q^{-\rho}, \dots, r_b q^{-\rho})$$

in the foregoing partition functions Z and Z' . Here p_1, p_2, \dots, p_a and r_1, r_2, \dots, r_b are two sets of parameters of these models, and $(p_1 q^{-\rho}, \dots, p_a q^{-\rho})$ and $(r_1 q^{-\rho}, \dots, r_b q^{-\rho})$ stand for the union of

$$p_i q^{-\rho} = (p_i q^{1/2}, p_i q^{3/2}, \dots, p_i q^{n-1/2}, \dots), \quad i = 1, \dots, a$$

and

$$r_j q^{-\rho} = (r_j q^{1/2}, r_j q^{3/2}, \dots, r_j q^{n-1/2}, \dots), \quad j = 1, \dots, b,$$

respectively. Since the Schur functions are symmetric functions, one can do such regrouping and reordering of arguments freely. The partition functions of the orbifold models are sums of these Boltzmann weights over all partitions:

$$Z_{a,b} = \sum_{\lambda \in \mathcal{P}} s_\lambda(p_1 q^{-\rho}, \dots, p_a q^{-\rho}) s_\lambda(r_1 q^{-\rho}, \dots, r_b q^{-\rho}) Q^{|\lambda|}, \quad (2.3)$$

$$Z'_{a,b} = \sum_{\lambda \in \mathcal{P}} s_\lambda(p_1 q^{-\rho}, \dots, p_a q^{-\rho}) s_{{}^t\lambda}(r_1 q^{-\rho}, \dots, r_b q^{-\rho}) Q^{|\lambda|}. \quad (2.4)$$

$Z'_{a,b}$ is essentially the Donaldson-Thomas partition function of local $\mathbb{CP}_{a,b}^1$ geometry presented by Bryan et al. [13].

By the Cauchy identities

$$\begin{aligned} \sum_{\lambda \in \mathcal{P}} s_\lambda(x_1, x_2, \dots) s_\lambda(y_1, y_2, \dots) &= \prod_{m,n=1}^{\infty} (1 - x_m y_n)^{-1}, \\ \sum_{\lambda \in \mathcal{P}} s_\lambda(x_1, x_2, \dots) s_{{}^t\lambda}(y_1, y_2, \dots) &= \prod_{m,n=1}^{\infty} (1 + x_m y_n), \end{aligned} \quad (2.5)$$

the partition functions turn into the product form

$$Z_{a,b} = \prod_{i=1}^a \prod_{j=1}^b M(p_i r_j Q, q), \quad (2.6)$$

$$Z'_{a,b} = \prod_{i=1}^a \prod_{j=1}^b M(-p_i r_j Q, q)^{-1}, \quad (2.7)$$

where $M(x, q)$ denotes the MacMahon function

$$M(x, q) = \prod_{n=1}^{\infty} (1 - xq^n)^{-n}.$$

Remark 1. Since homogeneity of the Schur functions imply the identities

$$\begin{aligned} s_{\lambda}(p_1 q^{-\rho}, \dots, p_a q^{-\rho}) &= s_{\lambda} \left(\frac{p_1}{p_a} q^{-\rho}, \dots, \frac{p_{a-1}}{p_a} q^{-\rho}, q^{-\rho} \right) p_a^{|\lambda|}, \\ s_{\lambda}(r_1 q^{-\rho}, \dots, r_b q^{-\rho}) &= s_{\lambda} \left(\frac{r_1}{r_b} q^{-\rho}, \dots, \frac{r_{b-1}}{r_b} q^{-\rho}, q^{-\rho} \right) r_b^{|\lambda|}, \end{aligned} \quad (2.8)$$

one can normalize the parameters of the models as

$$p_a = r_b = 1 \quad (2.9)$$

by replacing $Q \rightarrow Q(p_a r_b)^{-1}$.

Remark 2. Since

$$\begin{aligned} s_{\lambda}(q^{-\rho/a}) &= s_{\lambda}(q^{(1-a)/2a} q^{-\rho}, q^{(3-a)/2a} q^{-\rho}, \dots, q^{(a-1)/2a} q^{-\rho}), \\ s_{\lambda}(q^{-\rho/b}) &= s_{\lambda}(q^{(1-b)/2b} q^{-\rho}, q^{(3-b)/2b} q^{-\rho}, \dots, q^{(b-1)/2b} q^{-\rho}), \end{aligned}$$

these orbifold models reduce to the modifications

$$\tilde{Z}_{a,b} = \sum_{\lambda \in \mathcal{P}} s_{\lambda}(q^{\rho/a}) s_{\lambda}(q^{\rho/b}) Q^{|\lambda|}, \quad (2.10)$$

$$\tilde{Z}'_{a,b} = \sum_{\lambda \in \mathcal{P}} s_{\lambda}(q^{\rho/a}) s_{\lambda}(q^{\rho/b}) Q^{|\lambda|} \quad (2.11)$$

of the ordinary models (2.1) and (2.2) when the parameters are specialized as

$$\begin{aligned} p_1 &= q^{(1-a)/2a}, \quad p_2 = q^{(3-a)/2a}, \quad \dots \quad p_a = q^{(a-1)/2a}, \\ r_1 &= q^{(1-b)/2b}, \quad r_2 = q^{(3-b)/2b}, \quad \dots \quad r_b = q^{(b-1)/2b}. \end{aligned}$$

This is an unexpected link with our previous work on a melting crystal model with two q -parameters [22]. The previous model coincides with $\tilde{Z}_{a,b}$ when the q -parameters q_1, q_2 therein are specialized as

$$q_1 = q^{1/a}, \quad q_2 = q^{1/b}. \quad (2.12)$$

2.2 Fermionic formulation

The setup for the fermionic formulation [28] is the same as our earlier work [10, 11]:

- (i) Fourier modes ψ_n, ψ_n^* of the 2D free fermion fields $\psi(z), \psi^*(z)$ are labelled by integers $n \in \mathbb{Z}$, and satisfy the anti-commutation relations

$$\psi_m \psi_n^* + \psi_n^* \psi_m = \delta_{m+n,0}, \quad \psi_m \psi_n + \psi_n \psi_m = 0, \quad \psi_m^* \psi_n^* + \psi_n^* \psi_m^* = 0.$$

- (ii) The Fock space and its dual space are decomposed to the charge- s sectors for $s \in \mathbb{Z}$. An orthonormal basis of the charge- s sector is given by the ground states

$$\begin{aligned} \langle s| &= \langle -\infty | \cdots \psi_{s-1}^* \psi_s^*, \\ |s\rangle &= \psi_{-s} \psi_{-s+1} \cdots |-\infty\rangle \end{aligned}$$

and the excited states

$$\begin{aligned} \langle s, \lambda| &= \langle -\infty | \cdots \psi_{\lambda_i+s-i+1}^* \cdots \psi_{\lambda_2+s-1}^* \psi_{\lambda_1+s}^*, \\ |s, \lambda\rangle &= \psi_{-\lambda_1-s} \psi_{-\lambda_2-s+1} \cdots \psi_{-\lambda_i-s+i-1} \cdots |-\infty\rangle \end{aligned}$$

labelled by partitions $\lambda = (\lambda_i)_{i=1}^\infty \in \mathcal{P}$.

- (iii) The action of the fermion bilinears

$$\begin{aligned} L_0 &= \sum_{n \in \mathbb{Z}} n : \psi_{-n} \psi_n^* :, & W_0 &= \sum_{n \in \mathbb{Z}} n^2 : \psi_{-n} \psi_n^* :, \\ H_k &= \sum_{n \in \mathbb{Z}} q^{kn} : \psi_{-n} \psi_n^* :, & J_k &= \sum_{n \in \mathbb{Z}} : \psi_{-n} \psi_{n+k}^* : \end{aligned}$$

on the Fock space, where $: \ :$ denotes the normal ordering with respect to the vacuum states $\langle 0|, |0\rangle$, preserves the charge- s sector. J_0, L_0, W_0 and H_k 's are diagonal with respect to the basis $\{|\lambda, s\rangle\}_{\lambda \in \mathcal{P}}$:

$$\langle \lambda, s | J_0 | \mu, s \rangle = \delta_{\lambda\mu} s, \tag{2.13}$$

$$\langle \lambda, s | L_0 | \mu, s \rangle = \delta_{\lambda\mu} \left(|\lambda| + \frac{s(s+1)}{2} \right), \tag{2.14}$$

$$\langle \lambda, s | W_0 | \mu, s \rangle = \delta_{\lambda\mu} \left(\kappa(\lambda) + (2s+1)|\lambda| + \frac{s(s+1)(2s+1)}{6} \right), \tag{2.15}$$

$$\langle \lambda, s | H_k | \mu, s \rangle = \delta_{\lambda\mu} \Phi_k(\lambda, s), \tag{2.16}$$

where

$$\kappa(\lambda) = \sum_{i=1}^{\infty} \left(\left(\lambda_i - i + \frac{1}{2} \right)^2 - \left(-i + \frac{1}{2} \right)^2 \right)$$

and

$$\Phi_k(\lambda, s) = \sum_{i=1}^{\infty} (q^{k(\lambda_i + s - i + 1)} - q^{k(s - i + 1)}) + \frac{1 - q^{ks}}{1 - q^k} q^k.$$

(vi) The vertex operators [29, 30]

$$\Gamma_{\pm}(z) = \exp \left(\sum_{k=1}^{\infty} \frac{z^k}{k} J_{\pm k} \right), \quad \Gamma'_{\pm}(z) = \exp \left(- \sum_{k=1}^{\infty} \frac{(-z)^k}{k} J_{\pm k} \right)$$

and the multi-variable extensions

$$\Gamma_{\pm}(\mathbf{x}) = \prod_{i \geq 1} \Gamma_{\pm}(x_i), \quad \Gamma'_{\pm}(\mathbf{x}) = \prod_{i \geq 1} \Gamma'_{\pm}(x_i)$$

to $\mathbf{x} = (x_1, x_2, \dots)$, too, preserve the charge- s sector. The matrix elements become skew-Schur functions [27, 28]

$$\begin{aligned} \langle \lambda, s | \Gamma_{-}(\mathbf{x}) | \mu, s \rangle &= \langle \mu, s | \Gamma_{+}(\mathbf{x}) | \lambda, s \rangle = s_{\lambda/\mu}(\mathbf{x}), \\ \langle \lambda, s | \Gamma'_{-}(\mathbf{x}) | \mu, s \rangle &= \langle \mu, s | \Gamma'_{+}(\mathbf{x}) | \lambda, s \rangle = s_{\mathfrak{t}\lambda/\mathfrak{t}\mu}(\mathbf{x}). \end{aligned} \quad (2.17)$$

A fermionic expression of $Z_{a,b}$ can be derived as follows. Note that the special values of the Schur functions in the Boltzmann weights of $Z_{a,b}$ can be expressed in a fermionic form as

$$\begin{aligned} s_{\lambda}(p_1 q^{-\rho}, \dots, p_a q^{-\rho}) &= \langle 0 | \Gamma_{+}(p_1 q^{-\rho}) \cdots \Gamma_{+}(p_a q^{-\rho}) | \lambda, 0 \rangle, \\ s_{\lambda}(r_1 q^{-\rho}, \dots, r_b q^{-\rho}) &= \langle \lambda, 0 | \Gamma_{-}(r_b q^{-\rho}) \cdots \Gamma_{-}(r_1 q^{-\rho}) | 0 \rangle. \end{aligned}$$

Normalize p_a and r_b as shown in (2.8) and (2.9), and introduce new parameters P_1, P_2, \dots, P_{a-1} and R_1, R_2, \dots, R_{b-1} as

$$\begin{aligned} p_i &= P_i \cdots P_{a-1} \quad \text{for } i = 1, 2, \dots, a-1, & p_a &= 1, \\ r_j &= R_j \cdots R_{b-1} \quad \text{for } j = 1, 2, \dots, b-1, & r_b &= 1. \end{aligned} \quad (2.18)$$

Recall that setting $p_a = r_b = 1$ does not lead to loss of generality. The products of Γ_{\pm} 's in the expression of the Schur functions thereby become

$$\begin{aligned} &\Gamma_{+}(p_1 q^{-\rho}) \cdots \Gamma_{+}(p_{a-1} q^{-\rho}) \Gamma_{+}(q^{-\rho}) \\ &= \Gamma_{+}(P_1 \cdots P_{a-1} q^{-\rho}) \cdots \Gamma_{+}(P_{a-1} q^{-\rho}) \Gamma_{+}(q^{-\rho}) \\ &= (P_1 \cdots P_{a-1})^{-L_0} \Gamma_{+}(q^{-\rho}) P_1^{L_0} \Gamma_{+}(q^{-\rho}) P_2^{L_0} \cdots \Gamma_{+}(q^{-\rho}) P_{a-1}^{L_0} \Gamma_{+}(q^{-\rho}) \end{aligned}$$

and

$$\begin{aligned}
& \Gamma_-(q^{-\rho})\Gamma_-(r_{b-1}q^{-\rho})\cdots\Gamma_-(r_1q^{-\rho}) \\
&= \Gamma_-(q^{-\rho})\Gamma_-(R_{b-1}q^{-\rho})\cdots\Gamma_-(R_{b-1}\cdots R_2q^{-\rho})\Gamma_-(R_{b-1}\cdots R_1q^{-\rho}) \\
&= \Gamma_-(q^{-\rho})R_{b-1}^{L_0}\Gamma_-(q^{-\rho})\cdots R_2^{L_0}\Gamma_-(q^{-\rho})R_1^{L_0}\Gamma_-(q^{-\rho})(R_1\cdots R_{b-1})^{-L_0}.
\end{aligned}$$

The scaling property

$$u^{L_0}J_k u^{-L_0} = u^{-k}J_k, \quad k \in \mathbb{Z}, \quad (2.19)$$

of J_k 's and its consequences

$$\Gamma_+(vq^{-\rho})u^{L_0} = u^{L_0}\Gamma_+(uvq^{-\rho}), \quad u^{L_0}\Gamma_-(vq^{-\rho}) = \Gamma_-(uvq^{-\rho})u^{L_0} \quad (2.20)$$

have been used here. Since the leftmost factor $(P_1\cdots P_{a-1})^{-L_0}$ and the rightmost factor $(R_1\cdots R_{b-1})^{-L_0}$ disappear as they hit the vacuum vector, the fermionic expression of the special values of the Schur functions can be rewritten as

$$\begin{aligned}
& s_\lambda(p_1q^{-\rho}, \dots, p_{a-1}q^{-\rho}, q^{-\rho}) \\
&= \langle 0 | \Gamma_+(q^{-\rho})P_1^{L_0}\Gamma_+(q^{-\rho})P_2^{L_0}\cdots\Gamma_+(q^{-\rho})P_{a-1}^{L_0}\Gamma_+(q^{-\rho}) | \lambda, 0 \rangle
\end{aligned}$$

and

$$\begin{aligned}
& s_\lambda(r_1q^{-\rho}, \dots, r_{b-1}q^{-\rho}, q^{-\rho}) \\
&= \langle \lambda, 0 | \Gamma_-(q^{-\rho})R_{b-1}^{L_0}\Gamma_-(q^{-\rho})\cdots R_2^{L_0}\Gamma_-(q^{-\rho})R_1^{L_0}\Gamma_-(q^{-\rho}) | 0 \rangle.
\end{aligned}$$

Thus the partition function $Z_{a,b}$ can be cast into the fermionic expression

$$\begin{aligned}
Z_{a,b} &= \langle 0 | \Gamma_+(q^{-\rho})P_1^{L_0}\Gamma_+(q^{-\rho})P_2^{L_0}\cdots\Gamma_+(q^{-\rho})P_{a-1}^{L_0}\Gamma_+(q^{-\rho})Q^{L_0} \\
&\quad \times \Gamma_-(q^{-\rho})R_{b-1}^{L_0}\Gamma_-(q^{-\rho})\cdots R_2^{L_0}\Gamma_-(q^{-\rho})P_1^{L_0}\Gamma_-(q^{-\rho}) | 0 \rangle. \quad (2.21)
\end{aligned}$$

In the same way, using the primed versions

$$\begin{aligned}
& s_{\text{t}\lambda}(r_1q^{-\rho}, \dots, r_{b-1}q^{-\rho}, q^{-\rho}) \\
&= \langle \lambda, 0 | \Gamma'_-(q^{-\rho})R_{b-1}^{L_0}\Gamma'_-(q^{-\rho})\cdots R_2^{L_0}\Gamma'_-(q^{-\rho})R_1^{L_0}\Gamma'_-(q^{-\rho}) | 0 \rangle
\end{aligned}$$

and

$$\Gamma'_+(vq^{-\rho})u^{L_0} = u^{L_0}\Gamma'_+(uvq^{-\rho}), \quad u^{L_0}\Gamma'_-(vq^{-\rho}) = \Gamma'_-(vq^{-\rho})u^{L_0} \quad (2.22)$$

of the foregoing relations as well, one can derive the following fermionic expression of $Z'_{a,b}$:

$$\begin{aligned}
Z'_{a,b} &= \langle 0 | \Gamma_+(q^{-\rho})P_1^{L_0}\Gamma_+(q^{-\rho})P_2^{L_0}\cdots\Gamma_+(q^{-\rho})P_{a-1}^{L_0}\Gamma_+(q^{-\rho})Q^{L_0} \\
&\quad \times \Gamma'_-(q^{-\rho})R_{b-1}^{L_0}\Gamma'_-(q^{-\rho})\cdots R_2^{L_0}\Gamma'_-(q^{-\rho})P_1^{L_0}\Gamma'_-(q^{-\rho}) | 0 \rangle. \quad (2.23)
\end{aligned}$$

Note that the Γ_- 's in the expression of $Z_{a,b}$ are replaced by Γ'_- 's.

2.3 Deformations by external potentials

The deformed partition functions $Z_{a,b}(s, \mathbf{t})$ and $Z'_{a,b}(s, \mathbf{t}, \bar{\mathbf{t}})$, $\mathbf{t} = (t_1, t_2, \dots)$, $\bar{\mathbf{t}} = (\bar{t}_1, \bar{t}_2, \dots)$, are defined as

$$Z_{a,b}(s, \mathbf{t}) = \sum_{\lambda \in \mathcal{P}} s_\lambda(p_1 q^{-\rho}, \dots, p_a q^{-\rho}) s_\lambda(r_1 q^{-\rho}, \dots, r_b q^{-\rho}) \times Q^{|\lambda|+s(s+1)/2} e^{\Phi(\lambda, s, \mathbf{t})} \quad (2.24)$$

and

$$Z'_{a,b}(s, \mathbf{t}, \bar{\mathbf{t}}) = \sum_{\lambda \in \mathcal{P}} s_\lambda(p_1 q^{-\rho}, \dots, p_a q^{-\rho}) s_{\bar{\lambda}}(r_1 q^{-\rho}, \dots, r_b q^{-\rho}) \times Q^{|\lambda|+s(s+1)/2} e^{\Phi(\lambda, s, \mathbf{t}, \bar{\mathbf{t}})}, \quad (2.25)$$

where

$$\begin{aligned} \Phi(\lambda, s, \mathbf{t}) &= \sum_{k=1}^{\infty} t_k \Phi_k(\lambda, s), \\ \Phi(\lambda, s, \mathbf{t}, \bar{\mathbf{t}}) &= \sum_{k=1}^{\infty} t_k \Phi_k(\lambda, s) + \sum_{k=1}^{\infty} \bar{t}_k \Phi_{-k}(\lambda, s). \end{aligned}$$

Under the normalization of parameters as shown in (2.18),² these partition functions can be converted to a fermionic form as

$$\begin{aligned} Z_{a,b}(s, \mathbf{t}) &= \langle s | \Gamma_+(q^{-\rho}) P_1^{L_0} \Gamma_+(q^{-\rho}) P_2^{L_0} \dots \Gamma_+(q^{-\rho}) P_{a-1}^{L_0} \Gamma_+(q^{-\rho}) Q^{L_0} e^{H(\mathbf{t})} \\ &\quad \times \Gamma_-(q^{-\rho}) R_{b-1}^{L_0} \Gamma_-(q^{-\rho}) \dots R_2^{L_0} \Gamma_-(q^{-\rho}) P_1^{L_0} \Gamma_-(q^{-\rho}) | s \rangle \end{aligned} \quad (2.26)$$

and

$$\begin{aligned} Z'_{a,b}(s, \mathbf{t}, \bar{\mathbf{t}}) &= \langle s | \Gamma_+(q^{-\rho}) P_1^{L_0} \Gamma_+(q^{-\rho}) P_2^{L_0} \dots \Gamma_+(q^{-\rho}) P_{a-1}^{L_0} \Gamma_+(q^{-\rho}) Q^{L_0} e^{H(\mathbf{t}, \bar{\mathbf{t}})} \\ &\quad \times \Gamma'_-(q^{-\rho}) R_{b-1}^{L_0} \Gamma'_-(q^{-\rho}) \dots R_2^{L_0} \Gamma'_-(q^{-\rho}) P_1^{L_0} \Gamma'_-(q^{-\rho}) | s \rangle, \end{aligned} \quad (2.27)$$

where

$$H(\mathbf{t}) = \sum_{k=1}^{\infty} t_k H_k, \quad \bar{H}(\bar{\mathbf{t}}) = \sum_{k=1}^{\infty} \bar{t}_k H_{-k}, \quad H(\mathbf{t}, \bar{\mathbf{t}}) = H(\mathbf{t}) + \bar{H}(\bar{\mathbf{t}}).$$

²To absorb the extra factors $p_a^{|\lambda|}$ and $r_b^{|\lambda|}$ in (2.8) into redefinition of Q in this s -dependent setting, we have to correct these partition functions by the overall factor $(p_a r_b)^{s(s+1)/2}$. $Z_{a,b}(s, \mathbf{t})$ and $Z'_{a,b}(s, \mathbf{t}, \bar{\mathbf{t}})$ in the fermionic expression stand for those corrected partition functions.

3 Partition function as tau function: First orbifold model

3.1 Shift symmetries of quantum torus algebra

Let $V_n^{(k)}$, $k, m \in \mathbb{Z}$, denote the fermion bilinears

$$V_m^{(k)} = q^{-km/2} \sum_{n \in \mathbb{Z}} q^{kn} : \psi_{m-n} \psi_n^* :$$

that includes H_k 's and J_k 's as particular cases:

$$H_k = V_0^{(k)}, \quad J_k = V_k^{(0)}.$$

These fermion bilinears give a realization of (a central extension of) the 2D quantum torus algebra. Namely, they satisfy the commutation relations

$$\begin{aligned} [V_m^{(k)}, V_n^{(l)}] &= \begin{cases} (q^{(lm-kn)/2} - q^{(kn-lm)/2})(V_{m+n}^{(k+l)} - \delta_{m+n,0} \frac{q^{k+l}}{1-q^{k+l}}) & \text{if } k+l \neq 0, \\ (q^{-k(m+n)} - q^{k(m+n)})V_{m+n}^{(0)} + m\delta_{m+n,0} & \text{if } k+l = 0. \end{cases} \end{aligned} \quad (3.1)$$

The case where $k = l = 0$ reduces to the commutation relations

$$[J_m, J_n] = m\delta_{m+n,0} \quad (3.2)$$

of the $U(1)$ current algebra, whereas H_k 's are commutative.

The shift symmetries [10, 11, 1, 2] act on these fermion bilinears. Actually, there are three different types of shift symmetries:

(i) For $k > 0$ and $m \in \mathbb{Z}$,

$$\begin{aligned} \Gamma_+(q^{-\rho}) \left(V_m^{(k)} - \frac{q^k}{1-q^k} \delta_{m,0} \right) \Gamma_+(q^{-\rho})^{-1} \\ = (-1)^k \Gamma_-(q^{-\rho})^{-1} \left(V_{m+k}^{(k)} - \frac{q^k}{1-q^k} \delta_{m+k,0} \right) \Gamma_-(q^{-\rho}). \end{aligned} \quad (3.3)$$

(ii) For $k > 0$ and $m \in \mathbb{Z}$,

$$\begin{aligned} \Gamma'_+(q^{-\rho}) \left(V_m^{(-k)} + \frac{1}{1-q^k} \delta_{m,0} \right) \Gamma'_+(q^{-\rho})^{-1} \\ = \Gamma'_-(q^{-\rho})^{-1} \left(V_{m+k}^{(-k)} + \frac{1}{1-q^k} \delta_{m+k,0} \right) \Gamma'_-(q^{-\rho}). \end{aligned} \quad (3.4)$$

(iii) For $k, m \in \mathbb{Z}$,

$$q^{W_0/2} V_m^{(k)} q^{-W_0/2} = V_m^{(k-m)}. \quad (3.5)$$

These algebraic relations are used to convert the fermionic expression (2.26), (2.27) of the partition functions to tau functions. This comprises two parts:

1. In the first part, $e^{H(\mathbf{t})}$ and $e^{\bar{H}(\bar{\mathbf{t}})}$ are moved to the leftmost or rightmost position of the operator product. Note that $e^{H(\mathbf{t})}$ and $e^{\bar{H}(\bar{\mathbf{t}})}$ commute with Q^{L_0} . The first two sets (3.3) and (3.4) of the shift symmetries are used in this procedure.
2. When the first part is finished, $H_k = V_0^{(k)}$ and $H_{-k} = V_0^{(-k)}$ in $e^{H(\mathbf{t})}$ and $e^{\bar{H}(\bar{\mathbf{t}})}$ turn into $V_{ak}^{(k)}$ and $V_{-bk}^{(-k)}$. In the second part, $V_{ak}^{(k)}$ and $V_{-bk}^{(-k)}$ are transformed to $V_{ak}^{(0)} = J_{ak}$ and $V_{-bk}^{(0)} = J_{-bk}$. The operator $q^{W_0/2}$ in (3.5) is an avatar of “framing factors” in the topological vertex construction [14]. In the orbifold case, those framing factors are replaced by “fractional” ones [13]. We shall use such a variant of (3.5) in the subsequent calculations.

Although mostly parallel to the case of the previous models [10, 11, 1, 2], this procedure in the present setting becomes considerably complicated. This section is focused on the partition function $Z_{a,b}(s, \mathbf{t})$ of the first orbifold model. The partition function $Z'_{a,b}(s, \mathbf{t}, \bar{\mathbf{t}})$ of the second orbifold model is considered in the next section.

3.2 Moving $e^{H(\mathbf{t})}$ towards the left

Let us explain how to move

$$e^{H(\mathbf{t})} = \exp \left(\sum_{k=1}^{\infty} t_k V_k^{(0)} \right)$$

towards the left through the product of $P_i^{L_0} \Gamma_+(q^{-\rho})$'s in (2.26). This is a repetition of processes in which the exponential operator overtakes each unit $P_i^{L_0} \Gamma_+(q^{-\rho})$ of the operator product. In these processes, the exponent itself as well as the product of $P_i^{L_0} \Gamma_+(q^{-\rho})$'s are altered.

At the first stage where $\exp \left(\sum_{k=1}^{\infty} t_k V_0^{(k)} \right)$ overtakes $P_{a-1}^{L_0} \Gamma_+(q^{-\rho})$, $\Gamma_+(q^{-\rho})$ turns into $\Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho})$, and several c -number factors and an operator of the form $\Gamma_-(\cdots)^{-1}$ are generated.

Proposition 1.

$$\begin{aligned}
& \Gamma_+(q^{-\rho})P_1^{L_0} \cdots \Gamma_+(q^{-\rho})P_{a-1}^{L_0}\Gamma_+(q^{-\rho}) \exp \left(\sum_{k=1}^{\infty} t_k V_0^{(k)} \right) \\
&= \exp \left(\sum_{k=1}^{\infty} \frac{t_k q^k}{1 - q^k} \right) \prod_{i=1}^{a-1} M(P_i \cdots P_{a-1}, q)^{-1} \cdot \Gamma_-(P_1 \cdots P_{a-1} q^{-\rho})^{-1} \\
&\quad \times \Gamma_+(q^{-\rho})P_1^{L_0} \cdots \Gamma_+(q^{-\rho})P_{a-2}^{L_0}\Gamma_+(q^{-\rho}) \exp \left(\sum_{k=1}^{\infty} t'_k V_k^{(k)} \right) \\
&\quad \times P_{a-1}^{L_0} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}), \quad (3.6)
\end{aligned}$$

where

$$t'_k = (-1)^k P_{a-1}^{-k} t_k.$$

Proof. The proof consists of three steps. First, use the first shift symmetry (3.3) in the specialized form

$$\Gamma_+(q^{-\rho})V_0^{(k)}\Gamma_+(q^{-\rho})^{-1} - \frac{q^k}{1 - q^k} = (-1)^k \Gamma_-(q^{-\rho})^{-1} V_k^{(k)} \Gamma_-(q^{-\rho}).$$

This implies the operator identity

$$\begin{aligned}
& \Gamma_+(q^{-\rho}) \exp \left(\sum_{k=1}^{\infty} t_k V_0^{(k)} \right) = \exp \left(\sum_{k=1}^{\infty} \frac{t_k q^k}{1 - q^k} \right) \\
&\quad \times \Gamma_-(q^{-\rho})^{-1} \exp \left(\sum_{k=1}^{\infty} (-1)^k t_k V_k^{(k)} \right) \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}),
\end{aligned}$$

hence

$$\begin{aligned}
& \Gamma_+(q^{-\rho})P_1^{L_0} \cdots \Gamma_+(q^{-\rho})P_{a-1}^{L_0}\Gamma_+(q^{-\rho}) \exp \left(\sum_{k=1}^{\infty} t_k V_0^{(k)} \right) \\
&= \exp \left(\sum_{k=1}^{\infty} \frac{t_k q^k}{1 - q^k} \right) \Gamma_+(q^{-\rho})P_1^{L_0} \cdots \Gamma_+(q^{-\rho})P_{a-1}^{L_0} \\
&\quad \times \Gamma_-(q^{-\rho})^{-1} \exp \left(\sum_{k=1}^{\infty} (-1)^k t_k V_k^{(k)} \right) \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}).
\end{aligned}$$

Second, move the newly generated operator $\Gamma_-(q^{-\rho})^{-1}$ towards the left. This can be achieved by repeated use of the scaling property (2.20) and the commutation relation

$$\Gamma_+(uq^{-\rho})\Gamma_-(vq^{-\rho})^{-1} = M(uv, q)^{-1} \Gamma_-(vq^{-\rho})^{-1} \Gamma_+(uq^{-\rho}) \quad (3.7)$$

as follows:

$$\begin{aligned}
& \Gamma_+(q^{-\rho})P_1^{L_0} \cdots \Gamma_+(q^{-\rho})P_{a-1}^{L_0}\Gamma_-(q^{-\rho})^{-1} \\
&= \prod_{i=1}^{a-1} M(P_i \cdots P_{a-1}, q)^{-1} \cdot \Gamma_-(P_1 \cdots P_{a-1}q^{-\rho})^{-1} \\
&\quad \times \Gamma_+(q^{-\rho})P_1^{L_0} \cdots \Gamma_+(q^{-\rho})P_{a-2}^{L_0}\Gamma_+(q^{-\rho})P_{a-1}^{L_0}.
\end{aligned}$$

Lastly, use the scaling property

$$u^{L_0}V_m^{(k)}u^{-L_0} = u^{-m}V_m^{(k)} \quad (3.8)$$

of $V_m^{(k)}$'s to move $P_{a-1}^{L_0}$ to the right of the exponential operator as

$$P_{a-1}^{L_0} \exp \left(\sum_{k=1}^{\infty} (-1)^k t_k V_k^{(k)} \right) = \exp \left(\sum_{k=1}^{\infty} (-1)^k P_{a-1}^{-k} t_k V_k^{(k)} \right) P_{a-1}^{L_0}.$$

□

At the next stage, $\exp \left(\sum_{k=1}^{\infty} t'_k V_k^{(k)} \right)$ overtakes $P_{a-2}^{L_0}\Gamma_+(q^{-\rho})$. the calculations are fully parallel to the first stage. The first step is to use the operator identity

$$\Gamma_+(q^{-\rho}) \exp \left(\sum_{k=1}^{\infty} t'_k V_k^{(k)} \right) = \Gamma_-(q^{-\rho})^{-1} \exp \left(\sum_{k=1}^{\infty} (-1)^k t'_k V_{2k}^{(k)} \right) \Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})$$

that follows from the specialized shift symmetry

$$\Gamma_+(q^{-\rho})V_k^{(k)}\Gamma_+(q^{-\rho})^{-1} = (-1)^k \Gamma_-(q^{-\rho})^{-1}V_{2k}^{(k)}\Gamma_-(q^{-\rho}).$$

The second and third steps are to move $\Gamma_-(q^{-\rho})^{-1}$ towards the left end and $P_{a-2}^{L_0}$ to the right of the exponential operator. This leads to the algebraic relation

$$\begin{aligned}
& \Gamma_+(q^{-\rho})P_1^{L_0} \cdots \Gamma_+(q^{-\rho})P_{a-2}^{L_0}\Gamma_+(q^{-\rho}) \exp \left(\sum_{k=1}^{\infty} t'_k V_k^{(k)} \right) \\
&= \prod_{i=1}^{a-2} M(P_i \cdots P_{a-2}, q)^{-1} \cdot \Gamma_-(P_1 \cdots P_{a-2}q^{-\rho})^{-1} \\
&\quad \times \Gamma_+(q^{-\rho})P_1^{L_0} \cdots \Gamma_+(q^{-\rho})P_{a-3}^{L_0}\Gamma_+(q^{-\rho}) \exp \left(\sum_{k=1}^{\infty} t''_k V_{2k}^{(k)} \right) \\
&\quad \times P_{a-2}^{L_0}\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho}),
\end{aligned}$$

where

$$t_k'' = (-1)^k P_{a-2}^{-2k} t_k'.$$

The overtaking process can be repeated until the transformed exponential operator overtakes all $\Gamma_+(q^{-\rho})$'s and $P_i^{L_0}$'s. The net result reads as follows.

Proposition 2.

$$\begin{aligned} & \Gamma_+(q^{-\rho}) P_1^{L_0} \cdots \Gamma_+(q^{-\rho}) P_{a-1}^{L_0} \Gamma_+(q^{-\rho}) \exp \left(\sum_{k=1}^{\infty} t_k V_0^{(k)} \right) \\ &= \exp \left(\sum_{k=1}^{\infty} \frac{t_k q^k}{1 - q^k} \right) \prod_{1 \leq i \leq j \leq a-1} M(P_i \cdots P_j, q)^{-1} \\ & \quad \times \Gamma_-(q^{-\rho})^{-1} \prod_{j=1}^{a-1} \Gamma_-(P_1 \cdots P_j q^{-\rho})^{-1} \cdot \exp \left(\sum_{k=1}^{\infty} T_k V_{ak}^{(k)} \right) \\ & \quad \times \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) P_1^{L_0} \cdots \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) P_{a-1}^{L_0} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}), \quad (3.9) \end{aligned}$$

where

$$T_k = (-1)^{ak} P_1^{-(a-1)k} P_2^{-(a-2)k} \cdots P_{a-1}^{-k} t_k. \quad (3.10)$$

Thus, as $e^{H(\mathbf{t})}$ in (2.26) is transferred to the left end of the operator product, the terms $t_k H_k$ in $H(\mathbf{t})$ are transformed to $T_k V_{ak}^{(k)}$. One can thereby rewrite the left half of the fermionic expression (2.26) as

$$\begin{aligned} & \langle s | \Gamma_+(q^{-\rho}) P_1^{L_0} \cdots \Gamma_+(q^{-\rho}) P_{a-1}^{L_0} \Gamma_+(q^{-\rho}) Q^{L_0} e^{H(\mathbf{t})} \\ &= \exp \left(\sum_{k=1}^{\infty} \frac{t_k q^k}{1 - q^k} \right) \prod_{1 \leq i \leq j \leq a-1} M(P_i \cdots P_j, q)^{-1} \cdot \langle s | \exp \left(\sum_{k=1}^{\infty} T_k V_{ak}^{(k)} \right) \\ & \quad \times \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) P_1^{L_0} \cdots \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) P_{a-1}^{L_0} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) Q^{L_0}. \quad (3.11) \end{aligned}$$

Note that Γ_-^{-1} 's disappear because $\langle s | J_{-k} = 0$ for $k > 0$.

3.3 Converting $Z_{a,b}(s, \mathbf{t})$ to a tau function

To interpret $Z_{a,b}(s, \mathbf{t})$ as a tau function, $V_{ak}^{(k)}$'s in (3.11) have to be transformed to J_{ak} 's. This can be achieved by the following variant of the third shift symmetry (3.5).

Lemma 1. For $k \in \mathbb{Z}$,

$$q^{W_0/2a} V_{ak}^{(k)} q^{-W_0/2a} = J_{ak}. \quad (3.12)$$

Proof.

$$\begin{aligned}
q^{W_0/2a} V_{ak}^{(k)} q^{-W_0/2a} &= q^{-ak^2/2} \sum_{n \in \mathbb{Z}} q^{kn} q^{W_0/2a} : \psi_{ak-n} \psi_n^* : q^{-W_0/2a} \\
&= q^{-ak^2/2} \sum_{n \in \mathbb{Z}} q^{kn+(ak-n)^2/2a-n^2/2a} : \psi_{ak-n} \psi_n^* : \\
&= J_{ak}.
\end{aligned}$$

□

By (3.12), one can substitute

$$V_{ak}^{(k)} = q^{-W_0/2a} J_{ak} q^{W_0/2a}$$

in (3.11). Since $q^{-W_0/2a}$ turns into a c-number factor as

$$\langle s | q^{-W_0/2a} = q^{-s(s+1)(2s+1)/12a} \langle s |,$$

one can rewrite (3.11) as

$$\begin{aligned}
&\langle s | \Gamma_+(q^{-\rho}) P_1^{L_0} \cdots \Gamma_+(q^{-\rho}) P_{a-1}^{L_0} \Gamma_+(q^{-\rho}) Q^{L_0} e^{H(\mathbf{t})} \\
&= \exp \left(\sum_{k=1}^{\infty} \frac{t_k q^k}{1 - q^k} \right) q^{-s(s+1)(2s+1)/12a} \prod_{1 \leq i \leq j \leq a-1} M(P_i \cdots P_j, q)^{-1} \\
&\quad \times \langle s | \exp \left(\sum_{k=1}^{\infty} T_k J_{ak} \right) q^{W_0/2a} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) P_1^{L_0} \cdots \\
&\quad \times \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) P_{a-1}^{L_0} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) Q^{L_0}. \quad (3.13)
\end{aligned}$$

This expression of the left half of (2.26) shows that $Z_{a,b}(s, \mathbf{t})$ is essentially a tau function of the KP hierarchy. To see a link with the bi-graded Toda hierarchy, however, the right half of (2.26), too, has to be rewritten as follows.

Lemma 2.

$$\begin{aligned}
&\Gamma_-(q^{-\rho}) R_{b-1}^{L_0} \Gamma_-(q^{-\rho}) \cdots R_1^{L_0} \Gamma_-(q^{-\rho}) |s\rangle \\
&= \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) R_{b-1}^{L_0} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) \cdots R_1^{L_0} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) q^{W_0/2b} |s\rangle \\
&\quad \times q^{-s(s+1)(2s+1)/12b} \prod_{1 \leq i \leq j \leq b-1} M(R_i \cdots R_j, q)^{-1}. \quad (3.14)
\end{aligned}$$

Proof. Insert $\Gamma_+(q^{-\rho})\Gamma_+(q^{-\rho})^{-1}$ to the right of each of $\Gamma_-(q^{-\rho})$'s as

$$\begin{aligned} & \Gamma_-(q^{-\rho})R_{b-1}^{L_0}\Gamma_-(q^{-\rho})\cdots R_1^{L_0}\Gamma_-(q^{-\rho})|s\rangle \\ &= \Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})\Gamma_+(q^{-\rho})^{-1}R_{b-1}^{L_0}\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})\Gamma_+(q^{-\rho})^{-1}\cdots \\ & \quad \times R_1^{L_0}\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})\Gamma_+(q^{-\rho})^{-1}|s\rangle. \end{aligned}$$

The rightmost $\Gamma_+(q^{-\rho})^{-1}$ disappears as it hits $|s\rangle$. By the commutation relation (3.7), one can move the other $\Gamma_+(q^{-\rho})^{-1}$'s towards the right end of the operator product. The transformed operators $\Gamma_+(R_1\cdots R_jq^{-\rho})^{-1}$, $j = 1, \dots, b-1$, hit $|s\rangle$ and disappear. The outcome reads

$$\begin{aligned} & \Gamma_-(q^{-\rho})R_{b-1}^{L_0}\Gamma_-(q^{-\rho})\cdots R_1^{L_0}\Gamma_-(q^{-\rho})|s\rangle \\ &= \Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})R_{b-1}^{L_0}\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})\cdots R_1^{L_0}\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})|s\rangle \\ & \quad \times \prod_{1 \leq i \leq j \leq b-1} M(R_i \cdots R_j, q)^{-1}. \end{aligned}$$

Lastly, rewriting $|s\rangle$ as

$$|s\rangle = q^{W_0/2b}|s\rangle q^{-s(s+1)(2s+1)/12b},$$

one obtains (3.14). \square

The inner product of (3.13) and (3.14) yields the following final expression of $Z_{a,b}(s, \mathbf{t})$.

Proposition 3.

$$Z_{a,b}(s, \mathbf{t}) = f_{a,b}(s, \mathbf{t}) \langle s | \exp \left(\sum_{k=1}^{\infty} T_k J_{ak} \right) g | s \rangle, \quad (3.15)$$

where

$$\begin{aligned} f_{a,b}(s, \mathbf{t}) &= \exp \left(\sum_{k=1}^{\infty} \frac{t_k q^k}{1 - q^k} \right) q^{-s(s+1)(2s+1)(1/12a+1/12b)} \\ & \quad \times \prod_{1 \leq i \leq j \leq a-1} M(P_i \cdots P_j, q)^{-1} \cdot \prod_{1 \leq i \leq j \leq b-1} M(R_i \cdots R_j, q)^{-1} \end{aligned} \quad (3.16)$$

and

$$\begin{aligned} g &= q^{W_0/2a} \Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})P_1^{L_0} \cdots \Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})P_{a-1}^{L_0} \Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})Q^{L_0} \\ & \quad \times \Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})R_{b-1}^{L_0} \Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho}) \cdots R_1^{L_0} \Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})q^{W_0/2b}. \end{aligned} \quad (3.17)$$

Actually, one can derive another expression of $Z_{ab}(s, \mathbf{t})$ based on the same operator g as follows. This is a reason why we modify the right half of (2.26) as shown in (3.14).

Proposition 4.

$$Z_{a,b}(s, \mathbf{t}) = f_{a,b}(s, \mathbf{t}) \langle s | g \exp \left(\sum_{k=1}^{\infty} \bar{T}_k J_{-bk} \right) | s \rangle, \quad (3.18)$$

where

$$\bar{T}_k = (-1)^{bk} R_1^{-(b-1)k} R_2^{-(b-2)k} \cdots R_{b-1}^{-k} t_k. \quad (3.19)$$

Proof. Derivation of (3.18) is fully parallel to the case of (3.15) except that $e^{H(\mathbf{t})}$ is moved *towards the right*. Start from the specialization

$$\Gamma_+(q^{-\rho}) V_{-k}^{(k)} \Gamma_+(q^{-\rho})^{-1} = (-1)^k \left(\Gamma_-(q^{-\rho})^{-1} V_0^{(k)} \Gamma_-(q^{-\rho}) - \frac{q^k}{1-q^k} \right)$$

of the first shift symmetry (3.3). This implies that

$$\begin{aligned} \exp \left(\sum_{k=1}^{\infty} t_k V_0^{(k)} \right) \Gamma_-(q^{-\rho}) &= \left(\sum_{k=1}^{\infty} \frac{t_k q^k}{1-q^k} \right) \\ &\times \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) \exp \left(\sum_{k=1}^{\infty} (-1)^k t_k V_{-k}^{(k)} \right) \Gamma_+(q^{-\rho})^{-1}. \end{aligned}$$

Sending $\Gamma_+(q^{-\rho})^{-1}$ towards the right beyond the product of $R_i^{L_0}$'s and $\Gamma_-(q^{-\rho})$'s, one obtains the algebraic relation

$$\begin{aligned} &\exp \left(\sum_{k=1}^{\infty} t_k V_0^{(k)} \right) \Gamma_-(q^{-\rho}) R_{b-1}^{L_0} \Gamma_-(q^{-\rho}) \cdots R_1^{L_0} \Gamma_-(q^{-\rho}) \\ &= \exp \left(\sum_{k=1}^{\infty} \frac{t_k q^k}{1-q^k} \right) \prod_{i=1}^{a-1} M(R_i \cdots R_{b-1}, q)^{-1} \cdot \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) R_{b-1}^{L_0} \\ &\quad \times \exp \left(\sum_{k=1}^{\infty} \bar{t}'_k V_{-k}^{(k)} \right) \Gamma_-(q^{-\rho}) R_{b-2}^{L_0} \Gamma_-(q^{-\rho}) \cdots R_1^{L_0} \Gamma_-(q^{-\rho}) \\ &\quad \times \Gamma_+(R_1 \cdots R_{b-1} q^{-\rho})^{-1}, \end{aligned}$$

where

$$t'_k = (-1)^k R_{b-1}^{-k} t_k.$$

This relation amounts to (3.6). Repeating this transfer procedure leads to the following counterpart of (3.9):

$$\begin{aligned}
& \exp \left(\sum_{k=1}^{\infty} t_k V_0^{(k)} \right) \Gamma_-(q^{-\rho}) R_{b-1}^{L_0} \Gamma_-(q^{-\rho}) \cdots R_1^{L_0} \Gamma_-(q^{-\rho}) \\
&= \exp \left(\sum_{k=1}^{\infty} \frac{t_k q^k}{1 - q^k} \right) \prod_{1 \leq i \leq j \leq b-1} M(R_i \cdots R_j, q)^{-1} \\
&\times \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) R_{b-1}^{L_0} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) \cdots R_1^{L_0} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) \\
&\times \exp \left(\sum_{k=1}^{\infty} \bar{T}_k V_{-bk}^{(k)} \right) \Gamma_+(q^{-\rho})^{-1} \prod_{j=1}^{b-1} \Gamma_+(R_1 \cdots R_j q^{-\rho})^{-1}.
\end{aligned}$$

In much the same way, one can derive counterparts of (3.11), (3.13), (3.14). Equation (3.18) is an immediate consequence of these algebraic relations. \square

Equations (3.15) and (3.18) show that the partition function $Z_{a,b}(s, \mathbf{t})$ of the first orbifold model is related to the tau function

$$\tau(s, \mathbf{t}, \bar{\mathbf{t}}) = \langle s | \exp \left(\sum_{k=1}^{\infty} t_k J_k \right) g \exp \left(- \sum_{k=1}^{\infty} \bar{t}_k J_{-k} \right) | s \rangle \quad (3.20)$$

of the 2D Toda hierarchy [31] restricted to a subspace of the full time evolutions. The existence of the two expressions (3.15) and (3.18) implies the identity

$$\langle s | \exp \left(\sum_{k=1}^{\infty} T_k J_{ak} \right) g | s \rangle = \langle s | g \exp \left(\sum_{k=1}^{\infty} \bar{T}_k J_{-bk} \right) | s \rangle \quad (3.21)$$

of the restricted tau functions or, equivalently, the operator identities

$$\begin{aligned}
& (-1)^{ak} P_1^{-(a-1)k} P_2^{-(a-2)k} \cdots P_{a-1}^{-k} J_{ak} g \\
&= (-1)^{bk} R_1^{-(b-1)k} R_2^{-(b-2)k} \cdots R_{b-1}^{-k} g J_{-bk}, \quad k = 1, 2, \dots \quad (3.22)
\end{aligned}$$

These algebraic relations characterize tau functions of the bi-graded Toda hierarchy of bi-degree (a, b) [15, 16]. We shall reconfirm this fact in the Lax formalism.

In summary, we have proved the following relation to the bi-graded Toda hierarchy.

Theorem 1. *The partition function $Z_{a,b}(s, \mathbf{t})$ is related to the tau function (3.20) of the 2D Toda hierarchy in two ways as*

$$Z_{a,b}(s, \mathbf{t}) = f_{a,b}(s, \mathbf{t})\tau(s, \mathbf{T}, \mathbf{0}) = f_{a,b}(s, \mathbf{t})\tau(s, \mathbf{0}, -\bar{\mathbf{T}}), \quad (3.23)$$

where

$$\begin{aligned} \mathbf{T} &= (\underbrace{0, \dots, 0}_{a-1}, T_1, \underbrace{0, \dots, 0}_{a-1}, T_2, \dots, \underbrace{0, \dots, 0}_{a-1}, T_k, \dots), \\ \bar{\mathbf{T}} &= (\underbrace{0, \dots, 0}_{b-1}, \bar{T}_1, \underbrace{0, \dots, 0}_{b-1}, \bar{T}_2, \dots, \underbrace{0, \dots, 0}_{b-1}, \bar{T}_k, \dots). \end{aligned}$$

T_k 's and \bar{T}_k 's are obtained from t_k 's as shown in (3.10) and (3.19). The prefactor $f_{a,b}(s, \mathbf{t})$ is built from exponential and MacMahon functions as shown in (3.16). The generating operator (3.17) of the tau function satisfies the algebraic relations (3.22) that characterize the bi-graded Toda hierarchy of bi-degree (a, b) .

Remark 3. In the non-orbifold ($a = b = 1$) case, (3.21) reduces to the identity

$$\langle s | \exp \left(\sum_{k=1}^{\infty} t_k J_{ak} \right) g | s \rangle = \langle s | g \exp \left(\sum_{k=1}^{\infty} t_k J_{-bk} \right) | s \rangle,$$

which implies that $\tau(s, \mathbf{t}, \bar{\mathbf{t}})$ is a function of $\mathbf{t} - \bar{\mathbf{t}}$,

$$\tau(s, \mathbf{t}, \bar{\mathbf{t}}) = \tau(s, \mathbf{t} - \bar{\mathbf{t}}).$$

The reduced function $\tau(s, \mathbf{t})$ is a tau function of the 1D Toda hierarchy. Moreover, (3.23) takes the simpler form

$$Z_{1,1}(s, \mathbf{t}) = f_{1,1}(s, \mathbf{t})\tau(s, \iota(\mathbf{t}), \mathbf{0}) = f_{1,1}(s, \mathbf{t})\tau(s, \mathbf{0}, -\iota(\mathbf{t})),$$

where

$$\iota(\mathbf{t}) = (-t_1, t_2, -t_3, \dots, (-1)^k t_k, \dots),$$

thus both \mathbf{T} and $\bar{\mathbf{T}}$ coincide with $\iota(\mathbf{t})$.

Remark 4. In the orbifold model, the reduced time variables \mathbf{T} and $\bar{\mathbf{T}}$ of the tau function are full of *gaps*. Namely, the coupling constants t_k , $k = 1, 2, \dots$ of the orbifold model show up only at every a -steps in \mathbf{T} and every b -steps $\bar{\mathbf{T}}$, and the other components of \mathbf{T} and $\bar{\mathbf{T}}$ are set to 0. This is related to the structure of the Lax operators in the Lax formalism.

Remark 5. The operators $q^{W_0/2a}$ and $q^{W_0/2b}$ in the definition (3.17) of g are avatars of fractional framing factors in the orbifold topological vertex construction [13].

Remark 6. As mentioned in Remark 2, our previous melting crystal model with two q -parameters is a special case of the present orbifold model. This explains why the same bi-graded Toda hierarchy emerges therein when the q -parameters are specialized as shown in (2.12) [22].

4 Partition function as tau function: Second orbifold model

4.1 Moving $e^{\bar{H}(\bar{t})}$ towards the right

To convert the fermionic expression (2.27) of $Z'_{a,b}(s, \mathbf{t}, \bar{\mathbf{t}})$ to a tau function, we split $e^{H(\mathbf{t}, \bar{\mathbf{t}})}$ as

$$e^{H(\mathbf{t}, \bar{\mathbf{t}})} = e^{H(\mathbf{t})} e^{\bar{H}(\bar{\mathbf{t}})}$$

and move $e^{H(\mathbf{t})}$ and $e^{\bar{H}(\bar{\mathbf{t}})}$ towards the left and the right, respectively. Since the transfer procedure for $e^{H(\mathbf{t})}$ is exactly the same as in the case of $Z_{a,b}(s, \mathbf{t})$, let us explain how to move

$$e^{\bar{H}(\bar{\mathbf{t}})} = \exp \left(\sum_{k=1}^{\infty} \bar{t}_k V_0^{(-k)} \right)$$

towards the right. The whole procedure is parallel to the case of $e^{H(\mathbf{t})}$.

The following algebraic relation amounts to (3.6).

Proposition 5.

$$\begin{aligned} & \exp \left(\sum_{k=1}^{\infty} \bar{t}_k V_0^{(-k)} \right) \Gamma'_-(q^{-\rho}) R_{b-1}^{L_0} \Gamma'_-(q^{-\rho}) \cdots R_1^{L_0} \Gamma'_-(q^{-\rho}) \\ &= \exp \left(- \sum_{k=1}^{\infty} \frac{\bar{t}_k}{1 - q^k} \right) \prod_{i=1}^{a-1} M(R_1 \cdots R_j, q)^{-1} \cdot \Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\rho}) R_{b-1}^{L_0} \\ & \quad \times \exp \left(\sum_{k=1}^{\infty} \bar{t}'_k V_{-k}^{(-k)} \right) \Gamma'_-(q^{-\rho}) R_{b-2}^{L_0} \Gamma'_-(q^{-\rho}) \cdots R_1^{L_0} \Gamma'_-(q^{-\rho}) \\ & \quad \times \Gamma'_+(R_1 \cdots R_{b-1} q^{-\rho})^{-1}, \quad (4.1) \end{aligned}$$

where

$$\bar{t}'_k = R_{b-1}^{-k} \bar{t}_k.$$

Proof. The proof is parallel to that of (3.6). First, use the second shift symmetry (3.4) in the specialized form

$$\Gamma'_+(q^{-\rho}) V_{-k}^{(-k)} \Gamma'_+(q^{-\rho})^{-1} = \Gamma'_-(q^{-\rho})^{-1} V_0^{(-k)} \Gamma'_-(q^{-\rho}) + \frac{1}{1 - q^k}.$$

This implies the operator identity

$$\begin{aligned} \exp \left(\sum_{k=1}^{\infty} \bar{t}_k V_0^{(-k)} \right) \Gamma'_-(q^{-\rho}) &= \exp \left(- \sum_{k=1}^{\infty} \frac{\bar{t}_k}{1 - q^k} \right) \\ &\times \Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\rho}) \exp \left(\sum_{k=1}^{\infty} \bar{t}_k V_{-k}^{(-k)} \right) \Gamma'_+(q^{-\rho})^{-1}, \end{aligned}$$

hence,

$$\begin{aligned} &\exp \left(\sum_{k=1}^{\infty} \bar{t}_k V_0^{(-k)} \right) \Gamma'_-(q^{-\rho}) R_{b-1}^{L_0} \Gamma'_-(q^{-\rho}) \cdots R_1^{L_0} \Gamma'_-(q^{-\rho}) \\ &= \exp \left(- \sum_{k=1}^{\infty} \frac{\bar{t}_k}{1 - q^k} \right) \Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\rho}) \exp \left(\sum_{k=1}^{\infty} \bar{t}_k V_{-k}^{(-k)} \right) \Gamma'_+(q^{-\rho})^{-1} \\ &\quad \times R_{b-1}^{L_0} \Gamma'_-(q^{-\rho}) \cdots R_1^{L_0} \Gamma'_-(q^{-\rho}). \end{aligned}$$

Second, use the commutation relation

$$\Gamma'_-(uq^{-\rho})^{-1} \Gamma'_+(vq^{-\rho}) = M(uv, q)^{-1} \Gamma'_+(vq^{-\rho}) \Gamma'_-(uq^{-\rho})^{-1} \quad (4.2)$$

of the primed operators to move the newly generated operator $\Gamma'_+(q^{-\rho})^{-1}$ towards the right as

$$\begin{aligned} &\Gamma'_+(q^{-\rho})^{-1} R_{b-1}^{L_0} \Gamma'_-(q^{-\rho}) \cdots R_1 \Gamma'_-(q^{-\rho}) \\ &= \prod_{i=1}^{b-1} M(R_i \cdots R_{b-1}, q)^{-1} \cdot R_{b-1}^{L_0} \Gamma'_-(q^{-\rho}) \cdots R_1^{L_0} \Gamma'_-(q^{-\rho}) \\ &\quad \times \Gamma'_+(R_1 \cdots R_{b-1} q^{-\rho})^{-1}. \end{aligned}$$

Lastly, use the scaling property (3.8) to move $R_{b-1}^{L_0}$ to the left of the exponential operator as

$$\exp \left(\sum_{k=1}^{\infty} \bar{t}_k V_{-k}^{(-k)} \right) R_{b-1}^{L_0} = R_{b-1}^{L_0} \exp \left(\sum_{k=1}^{\infty} R_{b-1}^{-k} \bar{t}_k V_{-k}^{(-k)} \right).$$

□

Thus as $\exp \left(\sum_{k=1}^{\infty} \bar{t}_k V_{-k}^{(-k)} \right)$ overtakes $\Gamma'_-(q^{-\rho}) R_{b-1}^{L_0}$, $\bar{t}_k V_{-k}^{(-k)}$ and $\Gamma'_-(q^{-\rho})$ turns into $\bar{t}'_k V_{-k}^{(-k)}$ and $\Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\rho})$. Moreover, several c -number factors and an operator factor of the form $\Gamma'_+(\cdots)^{-1}$ are generated.

Repeating this overtaking process, one obtains the following counterpart of (3.9).

Proposition 6.

$$\begin{aligned}
& \exp \left(\sum_{k=1}^{\infty} \bar{t}_k V_0^{(-k)} \right) \Gamma'_-(q^{-\rho}) R_{b-1}^{L_0} \Gamma'_-(q^{-\rho}) \cdots R_1^{L_0} \Gamma'_-(q^{-\rho}) \\
&= \exp \left(- \sum_{k=1}^{\infty} \frac{\bar{t}_k}{1-q^k} \right) \prod_{1 \leq i \leq j \leq b-1} M(R_i \cdots R_j, q)^{-1} \\
&\quad \times \Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\rho}) R_{b-1}^{L_0} \Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\rho}) \cdots R_1^{L_0} \Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\rho}) \\
&\quad \times \exp \left(\sum_{k=1}^{\infty} \bar{T}_k V_{-bk}^{(-k)} \right) \Gamma'_+(q^{-\rho})^{-1} \prod_{j=1}^{b-1} \Gamma'_+(R_1 \cdots R_j e^{-\rho})^{-1}, \quad (4.3)
\end{aligned}$$

where

$$\bar{T}_k = R_1^{-(b-1)k} R_2^{-(b-2)k} \cdots R_{b-1}^{-k} \bar{t}_k. \quad (4.4)$$

4.2 Converting $Z'_{a,b}(s, t, \bar{t})$ to a tau function

We now proceed to converting (2.27) to a tau function. Since the left half of this expression is the same as that of (2.26), one can use (3.13) as it is.

As regards the right half of (2.27), the forgoing operator identity (4.3) implies that

$$\begin{aligned}
& e^{\bar{H}(\bar{t})} \Gamma'_-(q^{-\rho}) R_{b-1}^{L_0} \Gamma'_-(q^{-\rho}) \cdots R_1^{L_0} \Gamma'_-(q^{-\rho}) |s\rangle \\
&= \exp \left(- \sum_{k=1}^{\infty} \frac{\bar{t}_k}{1-q^k} \right) \prod_{1 \leq i \leq j \leq b-1} M(R_i \cdots R_j, q)^{-1} \\
&\quad \times \Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\rho}) R_{b-1}^{L_0} \Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\rho}) \cdots \\
&\quad \times R_1^{L_0} \Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\rho}) \exp \left(\sum_{k=1}^{\infty} \bar{T}_k V_{-bk}^{(-k)} \right) |s\rangle. \quad (4.5)
\end{aligned}$$

Moreover, one can use the identity

$$q^{W_0/2b} V_{-bk}^{(-k)} q^{-W_0/2b} = J_{-bk} \quad (4.6)$$

in place of (3.12) to rewrite the last part of (4.5) as

$$\begin{aligned}
& \exp \left(\sum_{k=1}^{\infty} \bar{T}_k V_{-bk}^{(-k)} \right) |s\rangle = q^{-W_0/2b} \exp \left(\sum_{k=1}^{\infty} \bar{T}_k J_{-bk} \right) q^{W_0/2b} |s\rangle \\
&= q^{s(s+1)(2s+1)/12b} q^{-W_0/2b} \exp \left(\sum_{k=1}^{\infty} \bar{T}_k J_{-bk} \right) |s\rangle.
\end{aligned}$$

Consequently, the right half of (2.27) can be expressed as

$$\begin{aligned}
& e^{\bar{H}(\bar{\mathbf{t}})} \Gamma'_-(q^{-\rho}) R_{b-1}^{L_0} \Gamma'_-(q^{-\rho}) \cdots R_1^{L_0} \Gamma'_-(q^{-\rho}) |s\rangle \\
&= \exp \left(- \sum_{k=1}^{\infty} \frac{\bar{t}_k}{1-q^k} \right) q^{s(s+1)(2s+1)/12b} \prod_{1 \leq i \leq j \leq b-1} M(R_i \cdots R_j, q)^{-1} \\
&\quad \times \Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\rho}) R_{b-1}^{L_0} \Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\rho}) \cdots \\
&\quad \times R_1^{L_0} \Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\rho}) q^{-W_0/2b} \exp \left(\sum_{k=1}^{\infty} \bar{T}_k J_{-bk} \right) |s\rangle. \quad (4.7)
\end{aligned}$$

The inner product of (3.13) and (4.7) yields the following expression of $Z'_{a,b}(s, \mathbf{t}, \bar{\mathbf{t}})$.

Proposition 7.

$$Z'_{a,b}(s, \mathbf{t}, \bar{\mathbf{t}}) = f'_{a,b}(s, \mathbf{t}, \bar{\mathbf{t}}) \langle s | \exp \left(\sum_{k=1}^{\infty} T_k J_{ak} \right) g' \exp \left(\sum_{k=1}^{\infty} \bar{T}_k J_{-bk} \right) |s\rangle, \quad (4.8)$$

where

$$\begin{aligned}
f'_{a,b}(s, \mathbf{t}, \bar{\mathbf{t}}) &= \exp \left(\sum_{k=1}^{\infty} \frac{t_k q^k - \bar{t}_k}{1-q^k} \right) q^{-s(s+1)(2s+1)(1/12a-1/12b)} \\
&\quad \times \prod_{1 \leq i \leq j \leq a-1} M(P_i \cdots P_j, q)^{-1} \cdot \prod_{1 \leq i \leq j \leq b-1} M(R_i \cdots R_j, q)^{-1} \quad (4.9)
\end{aligned}$$

and

$$\begin{aligned}
g' &= q^{W_0/2a} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) P_1^{L_0} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) \cdots P_{a-1}^{L_0} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) Q^{L_0} \\
&\quad \times \Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\rho}) R_{b-1}^{L_0} \Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\rho}) \cdots R_1^{L_0} \Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\rho}) q^{-W_0/2b}. \quad (4.10)
\end{aligned}$$

(4.8) shows that the partition function $Z'_{a,b}(s, \mathbf{t}, \bar{\mathbf{t}})$ is related to the tau function

$$\tau'(s, \mathbf{t}, \bar{\mathbf{t}}) = \langle s | \exp \left(\sum_{k=1}^{\infty} t_k J_k \right) g' \exp \left(- \sum_{k=1}^{\infty} \bar{t}_k J_{-k} \right) |s\rangle \quad (4.11)$$

of the 2D Toda hierarchy restricted to a subspace of the full time evolutions. Unlike the first orbifold model, the generating operator (4.10) seems to satisfy no particular algebraic relation like (3.22).

In summary, we have proved the following relation to the 2D Toda hierarchy.

Theorem 2. *The partition function $Z'_{a,b}(s, \mathbf{t}, \bar{\mathbf{t}})$ is related to the tau function (4.11) of the 2D Toda hierarchy as*

$$Z'_{a,b}(s, \mathbf{t}, \bar{\mathbf{t}}) = f'_{a,b}(s, \mathbf{t}, \bar{\mathbf{t}}) \tau'(\mathbf{T}, -\bar{\mathbf{T}}), \quad (4.12)$$

where

$$\begin{aligned} \mathbf{T} &= (\underbrace{0, \dots, 0}_{a-1}, T_1, \underbrace{0, \dots, 0}_{a-1}, T_2, \dots, \underbrace{0, \dots, 0}_{a-1}, T_k, \dots), \\ \bar{\mathbf{T}} &= (\underbrace{0, \dots, 0}_{b-1}, \bar{T}_1, \underbrace{0, \dots, 0}_{b-1}, \bar{T}_2, \dots, \underbrace{0, \dots, 0}_{b-1}, \bar{T}_k, \dots). \end{aligned}$$

T_k 's and \bar{T}_k 's are obtained from t_k 's and \bar{t}_k 's as shown in (3.10) and (4.4). The prefactor $f'_{a,b}(s, \mathbf{t})$ is built from exponential and MacMahon functions as shown in (4.9).

Remark 7. The reduced time variables \mathbf{T} and $\bar{\mathbf{T}}$ in (4.12) have the same “gaps” structure as in the case of (3.23). These gaps disappear in the non-orbifold ($a = b = 1$) case.

Remark 8. The structure of the generating operator (4.10) is reminiscent of that of the generating operator for open string amplitudes on generalized conifolds of the bubble type [3]. An essential difference is the emergence of the fractional framing factors $q^{W_0/2a}, q^{-W_0/2b}$ in place of the ordinary ones $q^{W_0/2}, q^{-W_0/2}$.

5 Initial values of dressing operators

5.1 Matrix representation of generating operators

The generating operators of Toda tau functions are non-degenerate elements of the Clifford group $\widehat{\text{GL}}(\infty)$ [28].³ Such a Clifford operator is given by the exponential $e^{\hat{X}}$ of a fermion bilinear

$$\hat{X} = \sum_{m,n \in \mathbb{Z}} x_{mn} \cdot \psi_{-m} \psi_n^*$$

or a product of such exponential operators. Fermion bilinears of this form are in-one-to-one correspondence with $\mathbb{Z} \times \mathbb{Z}$ matrices

$$X = \sum_{m,n \in \mathbb{Z}} x_{mn} E_{mn}, \quad E_{mn} = (\delta_{im} \delta_{jn})_{i,j \in \mathbb{Z}},$$

³To give regular solutions of the Toda hierarchy, they have to satisfy the extra condition $\langle s|g|s \rangle \neq 0$ as well for all $s \in \mathbb{Z}$.

and satisfy the commutation relation

$$[\hat{X}, \hat{Y}] = \widehat{[X, Y]} + c(X, Y),$$

where $c(X, Y)$ is a c-number cocycle of $\mathfrak{gl}(\infty)$. These fermion bilinears thus form a central extension $\widehat{\mathfrak{gl}}(\infty)$ of $\mathfrak{gl}(\infty)$. Clifford operators g correspond to $\mathbb{Z} \times \mathbb{Z}$ matrices $A = (a_{ij})_{i,j \in \mathbb{Z}}$ by the Bogoliubov transformation

$$g\psi_n g^{-1} = \sum_{m \in \mathbb{Z}} \psi_m a_{mn}$$

on the linear span of ψ_n 's. If g is given by the exponential $e^{\hat{X}}$ of a fermion bilinear \hat{X} , A becomes the exponential e^X of the associated matrix X .

Building blocks of the generating operators (3.17) and (4.10) can be translated to $\mathbb{Z} \times \mathbb{Z}$ matrices as follows. In matrix representation (for which we use the same notations as those of fermion bilinears), the fundamental fermion bilinears L_0, W_0, J_k can be expressed as

$$L_0 = \Delta, \quad W_0 = \Delta^2, \quad J_k = \Lambda^k, \quad (5.1)$$

where

$$\Delta = \sum_{n \in \mathbb{Z}} n E_{nn}, \quad \Lambda = \sum_{n \in \mathbb{Z}} E_{n, n+1}.$$

Note that Λ and Δ amount to the shift operator e^{∂_s} and the multiplication operator s on the spatial lattice \mathbb{Z} of the Toda hierarchy:

$$\Lambda \longleftrightarrow e^{\partial_s}, \quad \Delta \longleftrightarrow s. \quad (5.2)$$

$Q^{L_0}, P_i^{L_0}, R_j^{L_0}, q^{W_0/2a}$ and $q^{\pm W_0/2b}$ are Clifford operators, and their matrix representation can be obtained from the matrix representation of L_0, W_0 as

$$\begin{aligned} Q^{L_0} &= \sum_{n \in \mathbb{Z}} Q^n E_{nn}, & P_i^{L_0} &= \sum_{n \in \mathbb{Z}} P_i^n E_{nn}, & R_j^{L_0} &= \sum_{n \in \mathbb{Z}} R_j^n E_{nn}, \\ q^{W_0/2a} &= \sum_{n \in \mathbb{Z}} q^{n^2/2a} E_{nn}, & q^{\pm W_0/2b} &= \sum_{n \in \mathbb{Z}} q^{\pm n^2/2b} E_{nn}. \end{aligned} \quad (5.3)$$

In the same way, matrix representation of the one-variable vertex operators $\Gamma_{\pm}(z)$ and $\Gamma_{\pm}(z)$ can be calculated as

$$\begin{aligned} \Gamma_{\pm}(z) &= \exp \left(\sum_{k=1}^{\infty} \frac{z^k}{k} \Lambda^{\pm k} \right) = (1 - z\Lambda^{\pm 1})^{-1}, \\ \Gamma_{\pm}(z) &= \exp \left(- \sum_{k=1}^{\infty} \frac{(-z)^k}{k} \Lambda^{\pm k} \right) = 1 + z\Lambda^{\pm 1}. \end{aligned} \quad (5.4)$$

Consequently, $\Gamma_{\pm}(q^{-\rho})$ and $\Gamma'_{\pm}(q^{-\rho})$ become infinite products of the form

$$\begin{aligned}\Gamma_{\pm}(q^{-\rho}) &= \prod_{i=1}^{\infty} (1 - q^{i-1/2} \Lambda^{\pm 1})^{-1}, \\ \Gamma'_{\pm}(q^{-\rho}) &= \prod_{i=1}^{\infty} (1 + q^{i-1/2} \Lambda^{\pm 1}),\end{aligned}\tag{5.5}$$

which may be thought of as matrix-valued quantum dilogarithm [32, 33].

Thus the generating operators (3.17) and (4.10) correspond to the following matrices:

$$\begin{aligned}U &= q^{\Delta^2/2a} \Gamma_{-}(q^{-\rho}) \Gamma_{+}(q^{-\rho}) P_1^{\Delta} \Gamma_{-}(q^{-\rho}) \Gamma_{+}(q^{-\rho}) \cdots P_{a-1}^{\Delta} \Gamma_{-}(q^{-\rho}) \Gamma_{+}(q^{-\rho}) Q^{\Delta} \\ &\times \Gamma_{-}(q^{-\rho}) \Gamma_{+}(q^{-\rho}) R_{b-1}^{\Delta} \Gamma_{-}(q^{-\rho}) \Gamma_{+}(q^{-\rho}) \cdots R_1^{\Delta} \Gamma_{-}(q^{-\rho}) \Gamma_{+}(q^{-\rho}) q^{\Delta^2/2b},\end{aligned}\tag{5.6}$$

$$\begin{aligned}U' &= q^{\Delta^2/2a} \Gamma'_{-}(q^{-\rho}) \Gamma'_{+}(q^{-\rho}) P_1^{\Delta} \Gamma'_{-}(q^{-\rho}) \Gamma'_{+}(q^{-\rho}) \cdots P_{a-1}^{\Delta} \Gamma'_{-}(q^{-\rho}) \Gamma'_{+}(q^{-\rho}) Q^{\Delta} \\ &\times \Gamma'_{-}(q^{-\rho}) \Gamma'_{+}(q^{-\rho}) R_{b-1}^{\Delta} \Gamma'_{-}(q^{-\rho}) \Gamma'_{+}(q^{-\rho}) \cdots R_1^{\Delta} \Gamma'_{-}(q^{-\rho}) \Gamma'_{+}(q^{-\rho}) q^{-\Delta^2/2b}.\end{aligned}\tag{5.7}$$

5.2 Matrix factorization problem

Given a generating operator g of the tau function, the associated solution of the 2D Toda hierarchy in the Lax formalism can be captured by a matrix factorization problem [34, 35, 36] of the form

$$\exp\left(\sum_{k=1}^{\infty} t_k \Lambda^k\right) U \exp\left(-\sum_{k=1}^{\infty} \bar{t}_k \Lambda^{-k}\right) = W^{-1} \bar{W},\tag{5.8}$$

where U is a matrix representation of g , $W = W(\mathbf{t}, \bar{\mathbf{t}})$ is a lower triangular matrix with all diagonal elements being equal to 1, and $\bar{W} = \bar{W}(\mathbf{t}, \bar{\mathbf{t}})$ is an upper triangular matrix with all diagonal elements being non-zero, namely,

$$\begin{aligned}W &= 1 + w_1 \Lambda^{-1} + w_2 \Lambda^{-2} + \cdots, \\ \bar{W} &= \bar{w}_0 + \bar{w}_1 \Lambda + \bar{w}_2 \Lambda^2 + \cdots,\end{aligned}$$

where w_n 's and \bar{w}_n 's are diagonal matrices and \bar{w}_0 is invertible.

W and \bar{W} play the role of the dressing operators. They satisfy the Sato equations

$$\begin{aligned}\frac{\partial W}{\partial t_k} &= - (W \Lambda^k W^{-1})_{<0} W, & \frac{\partial W}{\partial \bar{t}_k} &= (\bar{W} \Lambda^{-k} \bar{W}^{-1})_{<0} W, \\ \frac{\partial \bar{W}}{\partial t_k} &= (W \Lambda^k W^{-1})_{\geq 0} \bar{W}, & \frac{\partial \bar{W}}{\partial \bar{t}_k} &= - (\bar{W} \Lambda^{-k} \bar{W}^{-1})_{\geq 0} \bar{W},\end{aligned}\tag{5.9}$$

where $(\)_{\geq 0}$ and $(\)_{< 0}$ denote the projection to the upper and strictly lower triangular parts, i.e.,

$$\left(\sum_{m,n} a_{mn} E_{mn} \right)_{\geq 0} = \sum_{m \leq n} a_{mn} E_{mn}, \quad \left(\sum_{m,n} a_{mn} E_{mn} \right)_{< 0} = \sum_{m > n} a_{mn} E_{mn}.$$

The Lax operators

$$L = \Lambda + u_1 + u_2 \Lambda^{-1} + \cdots, \\ \bar{L}^{-1} = \bar{u}_0 \Lambda^{-1} + \bar{u}_1 + \bar{u}_2 \Lambda + \cdots,$$

where u_n 's and \bar{u}_n 's are diagonal matrices, are obtained by “dressing” the shift matrices $\Lambda^{\pm 1}$ as

$$L = W \Lambda W^{-1}, \quad \bar{L}^{-1} = \bar{W} \Lambda^{-1} \bar{W}, \quad (5.10)$$

and satisfy the Lax equations

$$\frac{\partial L}{\partial t_k} = [B_k, L], \quad \frac{\partial L}{\partial \bar{t}_k} = [\bar{B}_k, L], \\ \frac{\partial \bar{L}^{-1}}{\partial t_k} = [B_k, \bar{L}^{-1}], \quad \frac{\partial \bar{L}^{-1}}{\partial \bar{t}_k} = [\bar{B}_k, \bar{L}^{-1}], \quad (5.11)$$

where

$$B_k = (L^k)_{\geq 0}, \quad \bar{B}_k = (\bar{L}^{-k})_{< 0}.$$

Solving the factorization problem (5.8) for bi-infinite matrices directly is an extremely tough problem unlike its analogue for finite or semi-infinite matrices [34]. In a sense, the fermionic construction of tau functions is an alternative approach to this issue [31]. In a cynical view, this approach simply converts a tough problem to another one, namely, to evaluating the right hand side of (3.20).

It is therefore remarkable that the factorization problem for the matrices (5.6) and (5.7) can be solved explicitly at least *at the initial time* $\mathbf{t} = \bar{\mathbf{t}} = \mathbf{0}$.

Proposition 8. *The generating matrices (5.6) and (5.7) can be factorized as*

$$U = W_{(0)}^{-1} \bar{W}_{(0)}, \quad U' = W'_{(0)}{}^{-1} \bar{W}'_{(0)}, \quad (5.12)$$

where

$$W_{(0)} = q^{\Delta^2/2a} \prod_{k=1}^{a+b} \Gamma_{-}(Q^{(k)} q^{-\rho})^{-1} \cdot q^{-\Delta^2/2a}, \\ \bar{W}_{(0)} = q^{\Delta^2/2a} \prod_{k=1}^{a+b} \Gamma_{+}(Q^{(k)-1} q^{-\rho}) \cdot (P_1 \cdots P_{a-1} Q R_{b-1} \cdots R_1)^{\Delta} q^{\Delta^2/2b}, \quad (5.13)$$

and

$$\begin{aligned}
W'_{(0)} &= q^{\Delta^2/2a} \prod_{i=1}^a \Gamma_-(Q^{(i)} q^{-\rho})^{-1} \cdot \prod_{j=1}^b \Gamma'_-(Q^{(a+j)} q^{-\rho})^{-1} \cdot q^{-\Delta^2/2a}, \\
\bar{W}'_{(0)} &= q^{\Delta^2/2a} \prod_{i=1}^a \Gamma_+(Q^{(i)-1} q^{-\rho}) \cdot \prod_{j=1}^b \Gamma'_+(Q^{(a+j)-1} q^{-\rho}) \\
&\quad \times (P_1 \cdots P_{a-1} Q R_{b-1} \cdots R_1)^\Delta q^{-\Delta^2/2b}.
\end{aligned} \tag{5.14}$$

The new constants $Q^{(i)}$'s are defined as

$$\begin{aligned}
Q^{(1)} &= 1, \quad Q^{(i)} = P_1 \cdots P_{i-1}, \quad i = 2, \dots, a, \\
Q^{(a+1)} &= P_1 \cdots P_{a-1} Q, \\
Q^{(a+j)} &= Q^{(a+1)} R_{b-1} \cdots R_{b-j+1}, \quad j = 2, \dots, b.
\end{aligned} \tag{5.15}$$

Proof. One can use the matrix version

$$\begin{aligned}
\Gamma_+(vq^{-\rho})u^\Delta &= u^\Delta \Gamma_+(uvq^{-\rho}), \quad u^\Delta \Gamma_-(vq^{-\rho}) = \Gamma_-(uvq^{-\rho})u^\Delta, \\
\Gamma'_+(vq^{-\rho})u^\Delta &= u^{L_0} \Gamma'_+(uvq^{-\rho}), \quad u^\Delta \Gamma'_-(vq^{-\rho}) = \Gamma'_-(uvq^{-\rho})u^\Delta
\end{aligned} \tag{5.16}$$

of (2.20) and (2.22) to collect P_i^Δ 's, R_j^Δ 's and Q^Δ in U and U' to the right end of the matrix product as

$$\begin{aligned}
U &= q^{\Delta^2/2a} \prod_{i=1}^a \Gamma_-(Q^{(i)} q^{-\rho}) \Gamma_+(Q^{(i)-1} q^{-\rho}) \\
&\quad \times \prod_{j=1}^b \Gamma_-(Q^{(a+j)} q^{-\rho}) \Gamma_+(Q^{(a+j)-1} q^{-\rho}) \cdot (P_1 \cdots P_{a-1} Q R_{b-1} \cdots R_1)^\Delta q^{\Delta^2/2b}
\end{aligned}$$

and

$$\begin{aligned}
U' &= q^{\Delta^2/2a} \prod_{i=1}^a \Gamma_-(Q^{(i)} q^{-\rho}) \Gamma_+(Q^{(i)-1} q^{-\rho}) \\
&\quad \times \prod_{j=1}^b \Gamma'_-(Q^{(a+j)} q^{-\rho}) \Gamma'_+(Q^{(a+j)-1} q^{-\rho}) \cdot (P_1 \cdots P_{a-1} Q R_{b-1} \cdots R_1)^\Delta q^{-\Delta^2/2b}.
\end{aligned}$$

Since Γ_\pm 's and Γ'_\pm 's are commutative, one can move Γ_- 's to the left side, Γ_+ 's and Γ'_+ 's to the right side, and insert $1 = q^{-\Delta^2/2a} \cdot q^{\Delta^2/2a}$ in the middle to achieve the factorization (5.12). \square

This result implies that $W_{(0)}, \bar{W}_{(0)}$ and $W'_{(0)}, \bar{W}'_{(0)}$ are nothing but the initial values $W|_{t=\bar{t}=0}, \bar{W}|_{t=\bar{t}=0}$ of the dressing operators for the special solutions of the 2D Toda hierarchy defined by the tau functions (3.20) and (4.11). These initial values, in turn, determine the initial values $L|_{t=\bar{t}=0}, \bar{L}|_{t=\bar{t}=0}$ of the Lax operators by the dressing relation (5.10). This eventually leads to a precise characterization of these special solutions as we shall show in the next section.

6 Structure of Lax operators

6.1 Initial values of Lax operators: First orbifold model

Let $L_{(0)}$ and $\bar{L}_{(0)}^{-1}$ denote the initial values of the Lax operators determined by U . As it turns out below, it is $L_{(0)}^a$ and $\bar{L}_{(0)}^{-b}$ rather than $L_{(0)}$ and $\bar{L}_{(0)}^{-1}$ themselves that play a fundamental role in the subsequent consideration. One can find their explicit form from (5.13).

The following are technical clues for these calculations.

Lemma 3.

$$\begin{aligned} q^{-\Delta^2/2a} \Lambda^a q^{\Delta^2/2a} &= q^{a/2} q^\Delta \Lambda^a, \\ u^\Delta q^{\Delta^2/2b} \Lambda^{-b} q^{-\Delta^2/2b} u^{-\Delta} &= u^b q^{-b/2} q^\Delta \Lambda^{-b}. \end{aligned} \quad (6.1)$$

Proof. Do straightforward calculations. \square

Lemma 4.

$$\begin{aligned} \Gamma_-(uq^{-\rho})^{-1} q^\Delta \Gamma_-(uq^{-\rho}) &= q^\Delta (1 - uq^{-1/2} \Lambda^{-1}), \\ \Gamma_+(u^{-1}q^{-\rho}) q^\Delta \Gamma_+(u^{-1}q^{-\rho})^{-1} &= q^\Delta (1 - u^{-1}q^{1/2} \Lambda). \end{aligned} \quad (6.2)$$

Proof. To derive the first identity, one can use the scaling property (5.16) and the infinite product form (5.5) to rewrite the left hand side as

$$\begin{aligned} \Gamma_-(uq^{-\rho})^{-1} q^\Delta \Gamma_-(uq^{-\rho}) &= q^\Delta \Gamma_-(uq^{-1}q^{-\rho})^{-1} \Gamma_-(uq^{-\rho}) \\ &= q^\Delta \prod_{i=1}^{\infty} (1 - uq^{i-3/2} \Lambda^{-1}) \cdot \prod_{i=1}^{\infty} (1 - uq^{i-1/2} \Lambda^{-1})^{-1} \\ &= q^\Delta (1 - uq^{-1/2} \Lambda^{-1}). \end{aligned}$$

In the same way, the second identity can be derived as

$$\begin{aligned} \Gamma_+(u^{-1}q^{-\rho}) q^\Delta \Gamma_+(u^{-1}q^{-\rho})^{-1} &= q^\Delta \Gamma_+(u^{-1}qq^{-\rho}) \Gamma_+(u^{-1}q^{-\rho})^{-1} \\ &= q^\Delta \prod_{i=1}^{\infty} (1 - u^{-1}q^{i+1/2} \Lambda)^{-1} \cdot \prod_{i=1}^{\infty} (1 - u^{-1}q^{i-1/2} \Lambda) \\ &= q^\Delta (1 - u^{-1}q^{1/2} \Lambda). \end{aligned}$$

□

Plugging (5.13) into the dressing relations

$$L_{(0)}^a = W_{(0)} \Lambda^a W_{(0)}^{-1}, \quad \bar{L}_{(0)}^{-b} = \bar{W}_{(0)} \Lambda^{-b} \bar{W}_{(0)}^{-1}$$

and applying (6.1), one can express $L_{(0)}^a$ and $\bar{L}_{(0)}^{-b}$ as

$$\begin{aligned} L_{(0)}^a &= q^{a/2} q^{\Delta^2/2a} \prod_{k=1}^{a+b} \Gamma_{-}(Q^{(k)} q^{-\rho})^{-1} \cdot q^{\Delta} \Lambda^a \prod_{k=1}^{a+b} \Gamma_{-}(Q^{(k)} q^{-\rho}) \cdot q^{-\Delta^2/2a} \\ &= q^{a/2} q^{\Delta^2/2a} \prod_{k=1}^{a+b} \Gamma_{-}(Q^{(k)} q^{-\rho})^{-1} \cdot q^{\Delta} \prod_{k=1}^{a+b} \Gamma_{-}(Q^{(k)} q^{-\rho}) \cdot \Lambda^a q^{-\Delta^2/2a} \end{aligned}$$

and

$$\begin{aligned} \bar{L}_{(0)}^{-b} &= (P_1 \cdots P_{a-1} Q R_{b-1} \cdots R_1)^b \\ &\quad \times q^{-b/2} q^{\Delta^2/2a} \prod_{k=1}^{a+b} \Gamma_{+}(Q^{(k)-1} q^{-\rho}) \cdot q^{\Delta} \Lambda^{-b} \prod_{k=1}^{a+b} \Gamma_{+}(Q^{(k)-1} q^{-\rho})^{-1} \cdot q^{-\Delta^2/2a} \\ &= (P_1 \cdots P_{a-1} Q R_{b-1} \cdots R_1)^b \\ &\quad \times q^{-b/2} q^{\Delta^2/2a} \prod_{k=1}^{a+b} \Gamma_{+}(Q^{(k)-1} q^{-\rho}) \cdot q^{\Delta} \prod_{k=1}^{a+b} \Gamma_{+}(Q^{(k)-1} q^{-\rho})^{-1} \cdot \Lambda^{-b} q^{-\Delta^2/2a}. \end{aligned}$$

By repeated use of (6.2), one can rewrite these expressions as

$$L_{(0)}^a = q^{a/2} q^{\Delta^2/2a} q^{\Delta} \prod_{k=1}^{a+b} (1 - Q^{(k)} q^{-1/2} \Lambda^{-1}) \cdot \Lambda^a q^{-\Delta^2/2a} \quad (6.3)$$

and

$$\begin{aligned} \bar{L}_{(0)}^{-b} &= (P_1 \cdots P_{a-1} Q R_{b-1} \cdots R_1)^b \\ &\quad \times q^{-b/2} q^{\Delta^2/2a} q^{\Delta} \prod_{k=1}^{a+b} (1 - Q^{(k)-1} q^{1/2} \Lambda) \cdot \Lambda^{-b} q^{-\Delta^2/2a}. \end{aligned} \quad (6.4)$$

Because of the identity

$$\prod_{k=1}^{a+b} (1 - Q^{(k)-1} q^{1/2} \Lambda) \cdot \Lambda^{-b} = \prod_{k=1}^{a+b} (-Q^{(k)-1} q^{1/2}) \cdot \prod_{k=1}^{a+b} (1 - Q^{(k)} q^{-1/2} \Lambda^{-1}) \cdot \Lambda^a,$$

$L_{(0)}^a$ and $\bar{L}_{(0)}^{-b}$ turn out to coincide with each other up to a constant factor:

$$\bar{L}_{(0)}^{-b} = (P_1 \cdots P_{a-1} Q R_{b-1} \cdots R_1)^b \prod_{k=1}^{a+b} (-Q^{(k)-1}) \cdot L_{(0)}^a.$$

Moreover, since

$$\prod_{k=1}^{a+b} (1 - Q^{(k)} q^{-1/2} \Lambda^{-1}) \cdot \Lambda^a = \prod_{i=1}^a (\Lambda - Q^{(i)} q^{-1/2}) \cdot \prod_{j=1}^b (1 - Q^{(a+j)} q^{-1/2} \Lambda^{-1}),$$

the foregoing results (6.3) and (6.4) can be restated in the following form.

Proposition 9. $L_{(0)}^a$ and $\bar{L}_{(0)}^{-b}$ can be factorized as

$$L_{(0)}^a = D^{-1} \bar{L}_{(0)}^{-b} = B_{(0)} C_{(0)}, \quad (6.5)$$

where

$$B_{(0)} = q^{a/2} q^{\Delta^2/2a} q^\Delta \prod_{i=1}^a (\Lambda - Q^{(i)} q^{-1/2}) \cdot q^{-\Delta^2/2a}, \quad (6.6)$$

$$C_{(0)} = q^{\Delta^2/2a} \prod_{j=1}^b (1 - Q^{(a+j)} q^{-1/2} \Lambda^{-1}) \cdot q^{-\Delta^2/2a}, \quad (6.7)$$

$$D = (P_1 \cdots P_{a-1} Q R_{b-1} \cdots R_1)^b \prod_{k=1}^{a+b} (-Q^{(k)-1}). \quad (6.8)$$

$B_{(0)}$ and $C_{(0)}$ are polynomials in $\Lambda^{\pm 1}$ of the form

$$\begin{aligned} B_{(0)} &= \Lambda^a + \beta_{1(0)} \Lambda^{a-1} + \cdots + \beta_{a(0)}, \\ C_{(0)} &= 1 + \gamma_{1(0)} \Lambda^{-1} + \cdots + \gamma_{b(0)} \Lambda^{-b}, \end{aligned} \quad (6.9)$$

where $\beta_{i(0)}$'s and $\gamma_{j(0)}$'s are diagonal matrices.

6.2 Initial values of Lax operators: Second orbifold model

Let $L'_{(0)}$ and $\bar{L}'_{(0)}{}^{-1}$ denote the initial values of the Lax operators determined by U' . In this case, too, $L_{(0)}^a$ and $\bar{L}_{(0)}^{-b}$ play a fundamental role. Let us derive their explicit form from (5.14).

Technical clues are the identities (6.1), (6.2) and their variants

$$u^\Delta q^{-\Delta^2/2b} \Lambda^{-b} q^{\Delta^2/2b} u^{-\Delta} = u^b q^{b/2} q^{-\Delta} \Lambda^{-b}, \quad (6.10)$$

$$\Gamma'_-(uq^{-\rho})^{-1} q^\Delta \Gamma'_-(uq^{-\rho}) = q^\Delta (1 + uq^{-1/2} \Lambda^{-1})^{-1} \quad (6.11)$$

and

$$\begin{aligned} \Gamma_+(u^{-1} q^{-\rho}) q^{-\Delta} \Gamma_+(u^{-1} q^{-\rho})^{-1} &= q^{-\Delta} (1 - u^{-1} q^{-1/2} \Lambda)^{-1}, \\ \Gamma'_+(u^{-1} q^{-\rho}) q^{-\Delta} \Gamma'_+(u^{-1} q^{-\rho})^{-1} &= q^{-\Delta} (1 + u^{-1} q^{-1/2} \Lambda). \end{aligned} \quad (6.12)$$

By the identities (6.1) and (6.10) and the dressing relations

$$L'_{(0)}{}^a = W'_{(0)} \Lambda^a W'_{(0)}{}^{-1}, \quad \bar{L}'_{(0)}{}^{-b} = \bar{W}'_{(0)} \Lambda^{-b} \bar{W}'_{(0)}{}^{-1},$$

$L'_{(0)}{}^a$ and $\bar{L}'_{(0)}{}^{-b}$ can be expressed as

$$\begin{aligned} L'_{(0)}{}^a &= q^{a/2} q^{\Delta^2/2a} \prod_{i=1}^a \Gamma_-(Q^{(i)} q^{-\rho})^{-1} \cdot \prod_{j=1}^b \Gamma'_-(Q^{(a+j)} q^{-\rho})^{-1} \\ &\quad \times q^\Delta \prod_{j=1}^b \Gamma'_-(Q^{(a+j)} q^{-\rho}) \cdot \prod_{i=1}^a \Gamma_-(Q^{(i)} q^{-\rho}) \cdot \Lambda^a q^{-\Delta^2/2a} \end{aligned}$$

and

$$\begin{aligned} \bar{L}'_{(0)}{}^{-b} &= (P_1 \cdots P_{a-1} Q R_{b-1} \cdots R_1)^b \\ &\quad \times q^{b/2} q^{\Delta^2/2a} \prod_{i=1}^a \Gamma_+(Q^{(i)-1} q^{-\rho}) \cdot \prod_{j=1}^b \Gamma'_+(Q^{(a+j)-1} q^{-\rho}) \\ &\quad \times q^{-\Delta} \prod_{j=1}^b \Gamma'_+(Q^{(a+j)-1} q^{-\rho})^{-1} \cdot \prod_{i=1}^a \Gamma_+(Q^{(i)-1} q^{-\rho})^{-1} \cdot \Lambda^{-b} q^{\Delta^2/2a}. \end{aligned}$$

By (6.2), (6.11) and (6.12), the right hand side can be simplified as

$$\begin{aligned} L'_{(0)}{}^a &= q^{a/2} q^{\Delta^2/2a} q^\Delta \prod_{i=1}^a (1 - Q^{(i)} q^{-1/2} \Lambda^{-1}) \\ &\quad \times \prod_{j=1}^b (1 + Q^{(a+j)} q^{-1/2} \Lambda^{-1})^{-1} \cdot \Lambda^a q^{-\Delta^2/2a} \end{aligned} \quad (6.13)$$

and

$$\begin{aligned}
\bar{L}'_{(0)}{}^{-b} &= (P_1 \cdots P_{a-1} Q R_{b-1} \cdots R_1)^b \\
&\quad \times q^{b/2} q^{\Delta^2/2a} q^{-\Delta} \prod_{i=1}^a (1 - Q^{(i)-1} q^{-1/2} \Lambda)^{-1} \\
&\quad \times \prod_{j=1}^b (1 + Q^{(a+j)-1} q^{-1/2} \Lambda) \cdot \Lambda^{-b} q^{-\Delta^2/2a}. \tag{6.14}
\end{aligned}$$

Since

$$\begin{aligned}
&q^{a/2} q^\Delta \prod_{i=1}^a (1 - Q^{(i)} q^{-1/2} \Lambda^{-1}) \cdot \prod_{j=1}^b (1 + Q^{(a+j)} q^{-1/2} \Lambda^{-1})^{-1} \cdot \Lambda^a \\
&= q^{a/2} q^\Delta \prod_{i=1}^a (\Lambda - Q^{(i)} q^{-1/2}) \cdot \prod_{j=1}^b (1 + Q^{(a+j)} q^{-1/2} \Lambda^{-1})^{-1}
\end{aligned}$$

and

$$\begin{aligned}
&q^{b/2} q^{-\Delta} \prod_{i=1}^a (1 - Q^{(i)-1} q^{-1/2} \Lambda)^{-1} \cdot \prod_{j=1}^b (1 + Q^{(a+j)-1} q^{-1/2} \Lambda) \cdot \Lambda^{-b} \\
&= q^{b/2} \prod_{i=1}^a (1 - Q^{(i)-1} q^{1/2} \Lambda)^{-1} \cdot \prod_{j=1}^b (1 + Q^{(a+j)-1} q^{1/2} \Lambda) \cdot \Lambda^{-b} q^{-b} q^{-\Delta} \\
&= \prod_{i=1}^a (-Q^{(i)}) \cdot \prod_{j=1}^b Q^{(a+j)-1} \cdot \prod_{j=1}^b (1 + Q^{(a+j)} q^{-1/2} \Lambda^{-1}) \\
&\quad \times \left(q^{a/2} q^\Delta \prod_{i=1}^a (\Lambda - Q^{(i)} q^{-1/2}) \right)^{-1},
\end{aligned}$$

the foregoing results (6.13) and (6.14) can be restated as follows.

Proposition 10. $L'_{(0)}{}^a$ and $\bar{L}'_{(0)}{}^{-b}$ can be factorized as

$$L'_{(0)}{}^a = B'_{(0)} C'_{(0)}{}^{-1}, \quad \bar{L}'_{(0)}{}^{-b} = D' C'_{(0)} B'_{(0)}{}^{-1}, \tag{6.15}$$

where

$$B'_{(0)} = q^{a/2} q^{\Delta^2/2a} q^\Delta \prod_{i=1}^a (\Lambda - Q^{(i)} q^{-1/2}) \cdot q^{-\Delta^2/2a}, \quad (6.16)$$

$$C'_{(0)} = q^{\Delta^2/2a} \prod_{j=1}^b (1 + Q^{(a+j)} q^{-1/2} \Lambda^{-1}) \cdot q^{-\Delta^2/2a}, \quad (6.17)$$

$$D' = (P_1 \cdots P_{a-1} Q R_{b-1} \cdots R_1)^b \prod_{i=1}^a (-Q^{(i)}) \cdot \prod_{j=1}^b Q^{(a+j)-1}. \quad (6.18)$$

$B'_{(0)}$ and $C'_{(0)}$ are polynomials in $\Lambda^{\pm 1}$ of the form

$$\begin{aligned} B'_{(0)} &= \Lambda^a + \beta'_{1(0)} \Lambda^{a-1} + \cdots + \beta'_{a(0)}, \\ C'_{(0)} &= 1 + \gamma'_{1(0)} \Lambda^{-1} + \cdots + \gamma'_{b(0)} \Lambda^{-b}, \end{aligned} \quad (6.19)$$

where $\beta'_{i(0)}$'s and $\gamma'_{j(0)}$'s are diagonal matrices. The inverse matrices of $B'_{(0)}$ and $C'_{(0)}$ are understood to be power series of $\Lambda^{\pm 1}$ of the form

$$\begin{aligned} B'_{(0)}{}^{-1} &= \beta'_{a(0)}{}^{-1} - \beta'_{a-1(0)} \beta'_{a(0)}{}^{-1} \Lambda \beta'_{a(0)}{}^{-1} + \cdots, \\ C'_{(0)}{}^{-1} &= 1 - \gamma'_{1(0)} \Lambda^{-1} + \cdots. \end{aligned} \quad (6.20)$$

6.3 Reductions of 2D Toda hierarchy

The factorized expressions (6.5) and (6.15) of the initial values of the Lax operators imply that these special solutions of the 2D Toda hierarchy belong to the following reductions.

- (i) *Bi-graded Toda hierarchy of bi-degree (a, b)* [15, 16]: This reduction is characterized by the algebraic relation

$$L^a = D^{-1} \bar{L}^{-b}, \quad (6.21)$$

where D is a non-zero constant. D can be normalized to $D = 1$ by rescaling \bar{t}_k 's. Under the reduction condition (6.21), both sides become a Laurent polynomial

$$\mathfrak{L} = \Lambda^a + \alpha_1 \Lambda^{a-1} + \cdots + \alpha_{a+b} \Lambda^{-b} \quad (6.22)$$

of bi-degree (a, b) in Λ with α_k 's being diagonal matrices. The Lax equations (5.11) of the 2D Toda hierarchy thereby reduce to the Lax equations

$$\frac{\partial \mathfrak{L}}{\partial t_k} = [B_k, \mathfrak{L}], \quad \frac{\partial \mathfrak{L}}{\partial \bar{t}_k} = [\bar{B}_k, \mathfrak{L}] \quad (6.23)$$

for the reduced Lax matrix \mathfrak{L} . L and \bar{L}^{-1} can be reconstructed from \mathfrak{L} as fractional powers

$$\begin{aligned}\mathfrak{L}^{1/a} &= \Lambda + u_1 + u_2\Lambda^{-1} + \cdots, \\ \mathfrak{L}^{1/b} &= \bar{u}_0\Lambda^{-1} + \bar{u}_1 + \bar{u}_2\Lambda + \cdots\end{aligned}$$

of two different (descending and ascending) types. A special case of this reduction is obtained by assuming the factorization

$$L^a = D^{-1}\bar{L}^{-b} = BC \quad (6.24)$$

of L^a and \bar{L}^{-b} , where B and C are polynomials in $\Lambda^{\pm 1}$ of the form

$$\begin{aligned}B &= \Lambda^a + \beta_1\Lambda^{a-1} + \beta_a, \\ C &= 1 + \gamma_1\Lambda^{-1} + \cdots + \gamma_b\Lambda^{-b}\end{aligned} \quad (6.25)$$

with β_i 's and γ_j 's being diagonal matrices. It is easy to see that the factorized form (6.24) of L^a and \bar{L}^{-b} is preserved by the time evolutions of the 2D Toda hierarchy. The proof is fully parallel to the case of the rational reductions [12]. Thus the initial values $L_{(0)}$ and $\bar{L}_{(0)}^{-1}$ determine a special solution of the bi-graded Toda hierarchy of bi-degree (a, b) .

- (ii) *Rational reduction of bi-degree (a, b)* [12]: This reduction is characterized by the factorization

$$L^a = BC^{-1}, \quad \bar{L}^{-b} = DCB^{-1}, \quad (6.26)$$

where B and C are matrices of the same form as (6.25) and D is a non-zero constant. D can be normalized to $D = 1$ by rescaling \bar{t}_k 's. The inverse matrices B^{-1} and C^{-1} are understood to be power series of $\Lambda^{\pm 1}$ of the form

$$\begin{aligned}B^{-1} &= \beta_a^{-1} - \beta_{a-1}\beta_a^{-1}\Lambda\beta_a^{-1} + \cdots, \\ C^{-1} &= 1 - \gamma_1\Lambda^{-1} + \cdots.\end{aligned} \quad (6.27)$$

L and \bar{L}^{-1} can be reconstructed from B and C as

$$\begin{aligned}L &= (BC^{-1})^{1/a} = \Lambda + u_1 + u_2\Lambda^{-1} + \cdots, \\ \bar{L}^{-1} &= (DCB^{-1})^{1/b} = \bar{u}_0\Lambda^{-1} + \bar{u}_1 + \bar{u}_2\Lambda + \cdots.\end{aligned}$$

As shown by Brini et al. [12], the factorized form (6.26) of L^a and \bar{L}^{-b} is preserved by the time evolutions of the 2D Toda hierarchy. The reduced

integrable hierarchy is a generalization of the Ablowitz-Ladik hierarchy [5, 6] or, equivalently, the relativistic Toda hierarchy [37, 38, 39], which amounts to the case where $a = b = 1$ [7]. Thus the initial values $L'_{(0)}$ and $\bar{L}'_{(0)}{}^{-1}$ determines a special solution of the rational reduction of bi-degree (a, b) .

We have thus arrived at the following conclusion.

Theorem 3.

- (i) *The generating matrix (5.6) determines a special solution of the 2D Toda hierarchy for which the Lax operators have the factorized form (6.24) at all time. The first orbifold model thus corresponds to a special solution of the bi-graded Toda hierarchy of bi-degree (a, b) .*
- (ii) *The generating matrix (5.7) determines a special solution of the 2D Toda hierarchy for which the Lax operators have the factorized form (6.26) at all time. The second orbifold model thus corresponds to a special solution of the rational reduction of bi-degree (a, b) .*

Remark 9. In the case of open string amplitudes on a generalized conifold of the bubble type [3], the Lax operators take yet another factorized form

$$L = B\Lambda^{1-N}C^{-1}, \quad \bar{L}^{-1} = DC\Lambda^{N-1}B^{-1}, \quad (6.28)$$

where B and C are polynomials in $\Lambda^{\pm 1}$ of the form

$$\begin{aligned} B &= \Lambda^N + \beta_1\Lambda^{N-1} + \cdots + \beta_N, \\ C &= 1 + \gamma_1\Lambda^{-1} + \cdots + \gamma_N\Lambda^{-N} \end{aligned} \quad (6.29)$$

with β_i 's and γ_i 's being diagonal matrices. An obvious difference between (6.26) and (6.28) is that the former, unlike the latter, contains the powers L^a, \bar{L}^{-b} of the Lax operators. This difference stems from the difference of the framing factors, namely, $q^{\Delta^2/2a}, q^{\Delta^2/2b}$ in the orbifold case and $q^{\Delta^2/2}$ in the ordinary ($a = b = 1$) case. The fractional framing factors $q^{\Delta^2/2a}, q^{\Delta^2/2b}$ yield the terms Λ^a, Λ^{-b} in (6.1) and (6.10). These powers of $\Lambda^{\pm 1}$ eventually turn into the powers L^a, \bar{L}^{-b} of L, \bar{L}^{-1} .

Acknowledgements

This work is partly supported by JSPS KAKENHI Grant No. 24540223 and No. 25400111.

References

- [1] K. Takasaki, Integrable structure of modified melting crystal model, arXiv:1208.4497 [math-ph].
- [2] K. Takasaki, Modified melting crystal model and Ablowitz-Ladik hierarchy, J. Phys. A: Math. Theor. **46** (2013), 245202, arXiv:1302.6129 [math-ph].
- [3] K. Takasaki, Generalized Ablowitz-Ladik hierarchy in topological string theory, J. Phys. A: Math. Theor. **47** (2014), 165201, arXiv:1312.7184 [math-ph].
- [4] A. Brini, The local Gromov-Witten theory of \mathbb{CP}^1 and integrable hierarchies, Commun. Math. Phys. **313** (2012), 571–605, arXiv:1002.0582 [math-ph].
- [5] M. J. Ablowitz and J. F. Ladik, Nonlinear differential-difference equations, J. Math. Phys. **16** (1975), 598–603.
- [6] V. E. Vekslerchik, Functional representation of the Ablowitz-Ladik hierarchy, J. Phys. A: Math. Gen. **31** (1998), 1087–1099, arXiv:solv-int/9707008.
- [7] A. Brini, G. Carlet and P. Rossi, Integrable hierarchies and the mirror model of local \mathbb{CP}^1 , Physica **D241** (2012), 2156–2167. arXiv:1105.4508 [math.AG].
- [8] K. Ueno and K. Takasaki, Toda lattice hierarchy, K. Okamoto (ed.), *Group Representations and Systems of Differential Equations*, Adv. Stud. Pure Math. vol. 4, Kinokuniya, Tokyo, 1984, pp. 1–95.
- [9] K. Takasaki and T. Takebe, Integrable hierarchies and dispersionless limit, Rev. Math. Phys. **7** (1995), 743–808, arXiv:hep-th/9405096.
- [10] T. Nakatsu and K. Takasaki, Melting crystal, quantum torus and Toda hierarchy, Commun. Math. Phys. **285** (2009), 445–468, arXiv:0710.5339 [hep-th].
- [11] T. Nakatsu and K. Takasaki, Integrable structure of melting crystal model with external potentials, M.-H. Saito, S. Hosono and K. Yoshioka (eds.), *New Defelopments in Algebraic Geometry, Integrable Systesm and Mirror Symmetry*, Adv. Stud. Pure Math. vol. 59, Mathematical Society of Japan, Tokyo, 2010, pp. 201–223, arXiv:0807.4970 [math-ph].

- [12] A. Brini, G. Carlet, S. Romano and P. Rossi, Rational reductions of the 2D-Toda hierarchy and mirror symmetry, arXiv:1401.5725 [math-ph].
- [13] J. Bryan, C. Cadman and B. Young, The orbifold topological vertex, Adv. Math. **229** (1) (2012), 531–595. arXiv:1008.4205 [math.AG].
- [14] M. Aganagic, A. Klemm, M. Mariño and C. Vafa, The topological vertex, Commun. Math. Phys. **254** (2005), 425–478, arXiv:hep-th/0305132.
- [15] B. A. Kuperschmidt, Discrete Lax equations and differential-difference calculus, Asterisque No. 123 (Société Mathématique de France, Paris, 1985).
- [16] G. Carlet, The extended bigraded Toda hierarchy, J. Phys. A: Math. Gen. **39** (2006), 9411–9435, arXiv:math-ph/0604024.
- [17] G. Carlet, B. Dubrovin and Y. Zhang, The extended Toda hierarchy, Moscow Math. J. **4** (2004), 313–332, arXiv:nlin/0306060.
- [18] T. E. Milanov, Hirota quadratic equations for the extended Toda hierarchy, Duke Math. J. **138** (2007), 161–178, arXiv:math.AG/0501336.
- [19] K. Takasaki, Two extensions of 1D Toda hierarchy, J. Phys. A: Math. Theor. **43** (2010), 434032, arXiv:1002.4688 [nlin.SI].
- [20] T. Milanov and H.-H. Tseng, The spaces of Laurent polynomials, Gromov-Witten theory of \mathbb{P}^1 -orbifolds, and integrable hierarchies, J. Reine Angew. Math. **622** (2008), 189–235, arXiv:math/0607012.
- [21] G. Carlet and J. van de Leur, Hirota equations for the extended bigraded Toda hierarchy and the total descendant potential of \mathbb{CP}^1 orbifolds, J. Phys. A: Math. Theor. **46** (2013), 405205, arXiv:1304.1632 [math-ph].
- [22] K. Takasaki, Integrable structure of melting crystal model with two q -parameters, J. Geom. Phys. **59** (2009), 1244–1257, arXiv:0903.2607 [math-ph].
- [23] A. Okounkov, N. Reshetikhin and C. Vafa, Quantum Calabi-Yau and classical crystals, P. Etingof, V. Retakh and I.M. Singer (eds.), *The unity of mathematics*, Progr. Math. vol. 244, Birkhäuser, 2006, pp. 597–618, arXiv:hep-th/0309208
- [24] T. Maeda, T. Nakatsu, K. Takasaki and T. Tamakoshi, Five-dimensional supersymmetric Yang-Mills theories and random plane partitions, JHEP **0503** (2005), 056, arXiv:hep-th/0412327.

- [25] J. Bryan and R. Pandharipande, The local Gromov-Witten theory of curves, *J. Amer. Math. Soc.* **21** (2008), 101–136, arXiv:math/0411037.
- [26] N. Caporaso, L. Griguolo, M. Mariño, S. Pasquetti and D. Seminara, Phase transitions, double-scaling limit, and topological strings, *Phys. Rev.* **D75** (2007), 046004, arXiv:hep-th/0606120.
- [27] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford University Press, 1995.
- [28] T. Miwa, M. Jimbo and E. Date, *Solitons: Differential equations, symmetries, and infinite-dimensional algebras*, Cambridge University Press, 2000.
- [29] A. Okounkov and N. Reshetikhin, Correlation function of Schur process with application to local geometry of a random 3-dimensional young diagram, *J. Amer. Math. Soc.* **16**, (2003), 581–603, arXiv:math.CO/0107056
- [30] J. Bryan and B. Young, Generating functions for coloured 3D Young diagrams and the Donaldson-Thomas invariants of orbifolds, *Duke Math. J.* **152** (2010), 115–153, arXiv:0802.3948 [math.CO].
- [31] T. Takebe, Representation theoretical meanings of the initial value problem for the Toda lattice hierarchy I, *Lett. Math. Phys.* **21** (1991), 77–84.
- [32] L. Faddeev and A. Yu. Volkov, Abelian current algebra and the Virasoro algebra on the lattice, *Phys. Lett.* **B315** (1993), 311–318, arXiv:hep-th/9307048.
- [33] L. D. Faddeev and R. M. Kashaev, Quantum dilogarithm, *Mod. Phys. Lett.* **A9** (1994), 427–434, arXiv:hep-th/9310070.
- [34] K. Takasaki, Initial value problem for the Toda lattice hierarchy, K. Okamoto (ed.), *Group Representations and Systems of Differential Equations*, Adv. Stud. Pure Math. vol. 4, Kinokuniya, Tokyo, 1984, pp. 136–163.
- [35] T. Nakatsu, K. Takasaki and S. Tsujimaru, Quantum and classical aspects of deformed $c = 1$ strings, *Nucl. Phys.* **B443** (1995), 1550–197, arXiv:hep-th/9501038.
- [36] K. Takasaki, Toda lattice hierarchy and generalized string equations, *Commun. Math. Phys.* **181** (1996), 131–156, arXiv:hep-th/9506089.

- [37] S. N. M. Ruijsenaars, Relativistic Toda systems, Commun. Math. Phys. **133** (1990), 217–247.
- [38] S. Kharchev, A. Mironov and A. Zhedanov, Faces of relativistic Toda chain, Int. J. Mod. Phys. **A12** (1997), 2675–2724, arXiv:hep-th/9606144.
- [39] Yu. B. Suris, A note on the integrable discretization of the non-linear Schrödinger equation, Inverse Problems **13** (1997), 1121–1136, arXiv:solv-int/9701010.